Mathematics Talent Reward Programme (MTRP), 2026

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Full Marks: 100 | Category: Junior | Solution Manual

Multiple Choice Questions $(4 \times 5 = 20)$

1. (B) $\boxed{45}$

   By Fermat's Little Theorem, So, we have:

   $$45^{2026} \equiv 1 \pmod{2027}$$

   Now,

   $$\begin{aligned} 45^{4053} &\equiv 45^{(2026\times 2)+1} \pmod{2027} \\ &\equiv (45^{2026})^2 \cdot 45^1 \pmod{2027} \\ &\equiv (1)^2 \cdot 45 \pmod{2027} \\ &\equiv 45 \pmod{2027} \end{aligned}$$

2. (B) $\boxed{\frac{\pi}{2}}$

   F is the midpoint of AB. Hence, OF is perpendicular to AB. $\angle OFB = \frac{\pi}{2}$. Hence, $\angle OFE = \frac{\pi}{2}$. Similarly, $\angle OGE = \frac{\pi}{2}$. Hence, in the quadrilateral $OFEG$, $\angle OFE + \angle OGE = \frac{\pi}{2} + \frac{\pi}{2} = \pi$. Therefore, $OFEG$ is a cyclic quadrilateral. Imagine the circle in which $OFEG$ is inscribed. Then $\angle OEG$ and $\angle OFG$ subtend the same arc of that circle. Hence,$\angle OEG = \angle OFG \implies \frac{\pi}{2} - \angle OEG = \frac{\pi}{2} - \angle OFG$. Since $\angle OFE = \frac{\pi}{2}$, $\frac{\pi}{2} - \angle OEG = \angle OFE - \angle OFG \implies \frac{\pi}{2} - \angle OEG = \angle GFE \implies \angle GFE + \angle OEG = \frac{\pi}{2} \cdots (1)$. Now consider triangle GME. Then, $\angle MEG + \angle MGE = \angle FME$, $\angle OEG = \angle MEG = \angle FME - \angle MGE$. But $\angle MGE = \angle FGE$, hence$\angle OEG = \angle FME - \angle FGE$.Substituting this value of $\angle OEG$ in equation (1), we obtain$\angle GFE + \angle FME - \angle FGE = \frac{\pi}{2}$.

3. (D) $\boxed{3}$

   Let $x^3 = 2^{a!} + 2^{b!} + 2^{c!}$. First see that no solutions exist when $a, b, c \geq 3$ (check $x \equiv 0 \pmod 3$ but $x^3 \equiv 3 \pmod 9$). A similar thing happens when two of them are ≥ 3 or only one of them is. So, $a, b, c \leq 2$ must hold. Just check that $2 + 2 + 4 = 8$ is the only way to get a perfect cube then. This can happen only when (a, b, c) is some permutation of $(1, 1, 2), (0, 1, 2)$ or $(0, 0, 2)$.

4. (C) $\boxed{146}$

Let the cost of a candy cane be c and the cost of a jingle bell be j, where c, j are positive integers and $c > j$. The total cost is $T = 12c + 10j$. Since both terms are even, T must be even. Hence 155 is not possible. Check

a) $12c + 10j = 146 \implies c = 8, j = 5$

b) $12c + 10j = 132 \implies c = 6, j = 6, \text{not satisfying } c > j$

c) $12c + 10j = 110 \implies c = 5, j = 5, \text{not satisfying } c > j$

5. (C) $\boxed{\binom{100}{3}}$

For any 3 points we can always find a hemisphere containing those 3 points.

Integer Type Questions $(4 \times 5 = 20)$

1. Answer: $\boxed{0}$

   Solution. Let N be the number of terms in the expansion of

   $$(a_1 + a_2 + \cdots + a_{12})^{24}$$

   in which every variable appears with an even power.

   To isolate terms with even powers, use the identity

   $$N = \frac{1}{2^{12}} \sum_{\varepsilon_i = \pm 1} (\varepsilon_1 a_1 + \varepsilon_2 a_2 + \cdots + \varepsilon_{12} a_{12})^{24}.$$

   Indeed, when expanding the right-hand side, any monomial $a_1^{k_1} \cdots a_{12}^{k_{12}}$ gets multiplied by

   $$\sum_{\varepsilon_i = \pm 1} \varepsilon_1^{k_1} \cdots \varepsilon_{12}^{k_{12}},$$

   which equals 2^{12} if all k_i are even and 0 otherwise.

   Now set $a_1 = a_2 = \cdots = a_{12} = 1$. Then

   $$\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{12} = 12 - 2j$$

   when exactly j of the ε_i are -1. There are $\binom{12}{j}$ such choices.

   Hence

   $$N = \frac{1}{2^{12}} \sum_{j=0}^{12} \binom{12}{j} (12 - 2j)^{24}.$$

   We need $N \pmod{13}$. Using Fermat's Little Theorem, since 13 is prime,

   $$x^{24} = (x^{12})^2 \equiv 1 \pmod{13} \quad \text{for } x \not\equiv 0 \pmod{13}.$$

   Now

   $$12 - 2j \equiv 0 \pmod{13} \quad \implies \quad j = 6.$$

   Thus for $j \neq 6$, $(12 - 2j)^{24} \equiv 1 \pmod{13}$, while for $j = 6$ the term is 0.

   Therefore

   $$\sum_{j=0}^{12} \binom{12}{j} (12 - 2j)^{24} \equiv \sum_{j \neq 6} \binom{12}{j} = 2^{12} - \binom{12}{6} \pmod{13}.$$

   Hence

   $$N \equiv \frac{2^{12} - \binom{12}{6}}{2^{12}} \pmod{13}.$$

   Since

   $$2^{12} = 4096 \equiv 1 \pmod{13},$$

   we get

   $$N \equiv 1 - \binom{12}{6} \pmod{13}.$$

Now

$$\binom{12}{6} = 924 \equiv 1 \pmod{13}.$$

Therefore

$$N \equiv 1 - 1 \equiv 0 \pmod{13}.$$

$$\boxed{0}$$

□

2. Answer: $\boxed{3}$

Solution. The time is 19 : 40 and after 31 mins, it shows 00 : 27. Clearly the last digit is faulty. The second last digit seems to be working fine. The first digit is also faulty (as it can't go back to 0). The actual time reads 09 : 49 and hence after 31 mins., it becomes 10 : 20 (in fact, it is the only solution). The answer is thus $1 + 0 + 2 + 0 = \boxed{3}$ □

3. Answer: $\boxed{33}$

Solution. The following is an example

$$ABCDABDACBDACDBACABDCBACDBACBDABC$$

For positive integers $k < 6$, the length of the shortest superpermutation, denoted $L(k)$, is given by the sum of the factorials from 1 to k:

$$L(k) = \sum_{i=1}^{k} i! = 1! + 2! + 3! + \cdots + k!$$

□

4. Answer: $\boxed{91}$.

Solution. Let $\triangle ABC$ be a triangle with sides $AB = 6$, $BC = 8$, and $CA = 10$. Since

$$6^2 + 8^2 = 36 + 64 = 100 = 10^2,$$

the triangle is right-angled with the right angle at B.

Let I be the incenter of $\triangle ABC$. The inradius is

$$r = \frac{a + b - c}{2} = \frac{6 + 8 - 10}{2} = 2.$$

Thus the distance from I to each side is 2. Reflecting I across a side places the image point at twice this distance from the side, so

$$IA' = IB' = IC' = 2r = 4.$$

Now consider triangle $A'B'C'$. Its area can be written as the sum of the areas of three triangles:

$$[A'B'C'] = [A'IC'] + [B'IA'] + [C'IB'].$$

First compute the area of $\triangle A'IC'$. The angle $\angle A'IC' = 180^\circ - A$, hence

$$[A'IC'] = \frac{1}{2} \cdot IA' \cdot IC' \cdot \sin(180^\circ - A) = \frac{1}{2} \cdot 4 \cdot 4 \cdot \sin A.$$

Since

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{BC}{CA} = \frac{8}{10} = \frac{4}{5},$$

we get

$$[A'IC'] = \frac{1}{2} \cdot 4 \cdot 4 \cdot \frac{4}{5} = \frac{32}{5}.$$

Similarly,

$$[B'IA'] = \frac{24}{5}, \qquad [C'IB'] = 8 = \frac{40}{5}.$$

Therefore,

$$[A'B'C'] = \frac{32}{5} + \frac{24}{5} + \frac{40}{5} = \frac{96}{5}.$$

Hence $m = 96$ and $n = 5$, giving

$$|m - n| = |96 - 5| = \boxed{91}.$$

Comment: This can also be solved using coordinate geometry. □

5. Answer: $\boxed{1}$

Solution. The given equation simplifies to

$$P(\tan\theta) = (\tan^{2026}\theta)\, P(\cot\theta) \qquad \forall\, \theta \in \left(0, \frac{\pi}{2}\right).$$

Replacing $\tan\theta$ with x, we obtain

$$P(x) = x^{2026} P\left(\frac{1}{x}\right) \qquad \forall\, x \in (0, \infty).$$

By the identity theorem, this equality holds for all $x \in \mathbb{R}$. Hence P satisfies the condition of a reciprocal polynomial (that is, 0 is not a root of P, and if r is a root of P, then $1/r$ is also a root of P).

Therefore the roots occur in reciprocal pairs, so the product of all the roots must be $\boxed{1}$. □

Subjective Type Questions $(10 \times 6 = 60)$

1. *Solution.* Let the first integer among the 8 consecutive positive integers be x. Then the integers are

$$x,\ x+1,\ x+2,\ x+3,\ x+4,\ x+5,\ x+6,\ x+7.$$

Given that

$$4p = x^2 + (x+1)^2 + (x+2)^2 + (x+3)^2 + (x+4)^2 + (x+5)^2 + (x+6)^2 + (x+7)^2.$$

Expanding,

$$4p = 8x^2 + 56x + 140 = 4(2x^2 + 14x + 35).$$

Dividing both sides by 4, we obtain

$$p = 2x^2 + 14x + 35.$$

Hence,

$$p + 1 = 2x^2 + 14x + 36 = 2(x^2 + 7x + 18).$$

Since p is a prime number greater than 3, we have

$$p \equiv 1 \pmod 6 \quad \text{or} \quad p \equiv 5 \pmod 6.$$

Case 1: $p \equiv 1 \pmod 6$**.**

Then

$$p + 1 \equiv 1 + 1 = 2 \pmod 6.$$

Hence,

$$2(x^2 + 7x + 18) \equiv 2 \pmod 6.$$

Dividing by 2,

$$x^2 + 7x + 18 \equiv 1 \pmod 3.$$

But since $18 \equiv 0 \pmod 3$, this becomes

$$x(x+7) \equiv 1 \pmod 3.$$

We now show that this is not possible. An integer modulo 3 can be congruent only to $0, 1,$ or 2.

a) If $x \equiv 0 \pmod 3$, then $x(x+7) \equiv 0 \pmod 3$.

b) If $x \equiv 1 \pmod 3$, then $x + 7 \equiv 2 \pmod 3$ and

$$x(x+7) \equiv 1 \cdot 2 = 2 \pmod 3.$$

c) If $x \equiv 2 \pmod 3$, then $x + 7 \equiv 0 \pmod 3$ and

$$x(x+7) \equiv 0 \pmod 3.$$

Thus $x(x+7)$ can be congruent only to 0 or 2 modulo 3, and never to 1. Hence this case is impossible.

Case 2: $p \equiv 5 \pmod 6$**.**

Then

$$p+1 \equiv 5+1 = 0 \pmod 6.$$

Hence

$$2(x^2+7x+18) \equiv 0 \pmod 6,$$

so

$$3 \mid (x^2+7x+18).$$

Since $18 \equiv 0 \pmod 3$, this implies

$$3 \mid x(x+7).$$

Also $2 \mid 2(x^2+7x+18)$ trivially. Therefore

$$12 \mid 2(x^2+7x+18) = p+1.$$

Hence

$$12 \mid (p+1).$$

□

2. *Solution.* $P(x)^2 - 1 = (P(x)-1)(P(x)+1)$, hence a root of $P(x)^2-1$, is a root of either $P(x)-1$ or $P(x)+1$(and not both).

   Suppose r is an integer satisfying $P(r) = 1$, and s is an integer satisftying $P(s) = -1$, then $r - s|1-(-1) \Rightarrow r-s|2$

   Hence, $|r-s| \leq 2$. Now suppose $r_1, \ldots r_k$ are distinct integer roots of $P(x)-1$ with $r_1 < r_2 \ldots < r_k$, and $s_1 \ldots s_l$ are distinct integer roots of $P(x)+1$ with $s_1 < s_2 \ldots < s_l$. Wlog, $r_1 < s_1$, $|r_1 - s_l| \leq 2$, hence if both $k, l > 0$, then $k+l \leq 3$. Hence, as $d \geq 1$, $3 \leq d+2$. Otherwise, atmost d many possible roots. □

3. *Solution.* Let a_n, b_n, c_n, d_n be the number of cards in the $1^{\text{st}}, 2^{\text{nd}}, 3^{\text{rd}}$ and 4^{th} piles after the n^{th} move. Let S_n be the net income of Ahan after the n^{th} move.

   Note that
$$T_n = \frac{a_n(a_n-1)+b_n(b_n-1)+c_n(c_n-1)+d_n(d_n-1)}{2} + S_n$$
   is a monovariant and increases by 1 in each step. Clearly,
$$T_{2026} - T_0 = 2026.$$
   Also,
$$T_{2026} - T_0 = S_{2026} - S_0 = S_{2026},$$
   since all piles contain the same number of cards at the end. Hence we get
$$S_{2026} = 2026.$$

Therefore, no matter what sequence of moves Ahan follows, he is guaranteed to have a net income of 2026 rupees at the end of the game.

□

4. *Solution.* We use induction on n. The theorem is obvious if $n = 1$ or 2. Suppose it is true for $1, 2, \ldots, n-1$, where $n \geq 3$.

Then we can restrict attention to odd n, since otherwise

$$\prod_{p \leq n} p = \prod_{p \leq n-1} p < 4^{n-1} < 4^{n}.$$

Thus we may write $n = 2m+1$. From its definition, the binomial coefficient

$$\binom{2m+1}{m} = \frac{(2m+1)!}{m!(m+1)!}$$

is divisible by every prime p with $m+2 \leq p \leq 2m+1$.
Hence

$$\prod_{p \leq 2m+1} p \leq \binom{2m+1}{m} \times \prod_{p \leq m+1} p < \binom{2m+1}{m} 4^{m+1}.$$

But the numbers

$$\binom{2m+1}{m} \quad \text{and} \quad \binom{2m+1}{m+1}$$

are equal, and both occur in the expansion of $(1+1)^{2m+1}$, so that

$$\binom{2m+1}{m} \leq \frac{1}{2} 2^{2m+1} = 4^{m}.$$

Therefore

$$\prod_{p \leq 2m+1} p < 4^{m} \cdot 4^{m+1} = 4^{2m+1}.$$

The theorem follows by induction on n. □

5. *Solution.* Take $a_{n+1} = 0$. Since $a_k - a_{k+1} \geq 0$ for $1 \leq k \leq n$, we have

$$(a_k - a_{k+1}) \sum_{j=1}^{k} a_j \leq (a_k - a_{k+1}) \sum_{j=1}^{k} b_j,$$

for $1 \leq k \leq n$.

Summing this over k, we obtain

$$a_1^2 + a_2^2 + \cdots + a_n^2 \leq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Applying the Cauchy–Schwarz inequality to the right-hand side, we get

$$a_1^2 + a_2^2 + \cdots + a_n^2 \leq \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)^{1/2} \left(b_1^2 + b_2^2 + \cdots + b_n^2\right)^{1/2}.$$

This simplifies to

$$a_1^2 + a_2^2 + \cdots + a_n^2 \leq b_1^2 + b_2^2 + \cdots + b_n^2.$$

□

6. *Solution.* **(a) Three trees.**

Suppose three trees are not on the same straight line. To protect all three, draw a sufficiently large circle containing them. To protect none, draw a small circle far away from the trees. To protect only one tree, draw a small circle around that tree. To protect exactly two trees, draw a circle passing through those two trees while leaving the third tree outside.

Thus every possible choice of protected and unprotected trees can be realized by a circle. Hence any three non-collinear trees can always be separated by a circle.

(b) Four trees.

Consider four trees located at the corners of a square. Suppose the wizard wishes to protect two opposite trees but not the other two. Any circle that contains the two opposite trees must also contain at least one of the remaining trees. Hence it is impossible to draw a circle that contains exactly those two opposite trees while excluding the other two.

Therefore such an arrangement cannot be achieved by a circle, and four trees cannot always be separated according to an arbitrary choice. □

End of Paper