Mathematics Talent Reward Programme (MTRP), 2026

Disclaimer: MTRP is an initiative of the students of Indian Statistical Institute, Kolkata, as a part of their annual techno-cultural-sports fest, Integration, to bolster the love for mathematics among all. This is to clarify that **Indian Statistical Institute itself does NOT conduct this event.**

Full Marks: 100 Category: Senior Time: 3 hours

Multiple Choice Questions $(3 \times 8 = 24)$

*Each question carries **3 points**. Only one out of the four options is correct. Correct answers without proper justification will not receive full marks.*

1. The number of unordered triplets (a, b, c) of non-negative integers such that $2^{a!} + 2^{b!} + 2^{c!}$ is a perfect cube is

   (A) 12 **(B)** 6 **(C)** 9 **(D)** 3

2. Let $Q(x)$ be a polynomial of degree 99 with real coefficients. Suppose that for every $x \in \mathbb{R}$, the polynomial satisfies $Q(1 + e^x) + Q(1 - e^x) = Q(2)$. Find the sum of roots of $Q'(x)$, the derivative of $Q(x)$.

   (A) 78 **(B)** 98 **(C)** 104 **(D)** 63

3. Let $\theta_1, \theta_2, \ldots, \theta_{2026}$ be non-negative reals such that $\theta_1 + \cdots + \theta_{2026} = 2\pi$. Let $z_1 = \cos\theta_1 + i\sin\theta_1$ and $z_{k+1} = (\cos\theta_{k+1} + i\sin\theta_{k+1})\, z_k$. Which of the following statements is true?

   (A) $|z_2^2 - z_1^2| + |z_3^2 - z_2^2| + \cdots + |z_{2026}^2 - z_1^2| \leq 4\pi$, where equality can occur

   (B) $|z_2^2 - z_1^2| + |z_3^2 - z_2^2| + \cdots + |z_{2026}^2 - z_1^2| < 4\pi$

   (C) $|z_2 - z_1| + |z_3 - z_2| + \cdots + |z_{2026} - z_1| < 2\pi$

   (D) Both (B) and (C) are true

4. Let $ABCD$ be a parallelogram, and E and F be two points on AB and AD, respectively, such that $BE : EA = 1 : 2$ and $AF : FD = 2 : 1$. Let BF and DE meet at K. If $AC = 10$, find CK.

   (A) 6 **(B)** 8 **(C)** $\frac{20}{3}$ **(D)** $\frac{21}{4}$

5. We say that an integer n is *square-free* if for every prime number p, p^2 does not divide n. Let $f(n)$ be the sum of reciprocals of all the positive integer divisors of n. (For example, $f(4) = 1 + 1/2 + 1/4 = 7/4$.) We define n to be an *amazing integer* if $f(n) = 2$. How many square-free amazing integers are there?

   (A) 1 **(B)** 2

   (C) more than 2 but finitely many **(D)** infinitely many

6. Let x denote the sum of digits of a 2026-digit positive integer that is divisible by 9. Let the sum of digits of x be y and the sum of digits of y be z. What is the maximum possible value of z?

(A) 9 **(B)** 18 **(C)** 27 **(D)** 36

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $\int_{\theta}^{\theta+1} f(x)dx = 0$ for every $\theta \in \mathbb{R}$. Let $g : [a, b] \to \mathbb{R}$ be continuously differentiable and $\int_a^b g(x)dx = \int_a^b g'(x)dx = 0$. Which of the following is NOT true?

(A) If $g(x)$ has a root in (a, b), it must have at least two roots in (a, b)

(B) If $g(x)$ has a root in $[a, b]$, it must have at least two roots in $[a, b]$

(C) $f(x) = 0$ is not the only continuous solution

(D) If $f(x)$ is continuous, it must be periodic

8. Let p, q, r be prime numbers such that $p^2 + q^2 + r^2 = p^3$. Find the average value of pqr over all possible values of the ordered triplet (p, q, r).

(A) 3 **(B)** 27 **(C)** 150 **(D)** None of these

Integer Type Questions $(4 \times 4 = 16)$

Each question carries 4 ***points****. The answer to each question is a non-negative integer.*
Correct answers without proper justification will not receive full marks.

1. Consider the 2-dimensional plane. Ozu the Ant is standing at $(0, 0)$ and wants to reach $(20, 20)$, using only one unit upward or rightward step at a time. Find the average of the area enclosed by the path taken by Ozu, the x-axis and the line $x = 20$. (Average is taken over all possible such paths.)

2. Suppose that $f : \mathbb{N} \to \mathbb{R}$ is a function satisfying, for every $n \geq 1$,

$$f(1) + f(2) + \cdots + f(n) = n^2 f(n).$$

If $f(1) = 2027$, find the value of $1/f(2026)$.

3. The longtime caretaker of the Overlook Hotel, Mr. Jack Torrance, has built a strange blending machine. When two bottles of glowing liquid with strengths x and y are placed into the machine, it produces a new bottle whose strength is $f(x, y)$. Jack does not reveal how the machine works, but he shares one rule: if you blend two bottles and then blend the result with a third bottle of strength z, the final strength always satisfies

$$f(f(x, y), z) = \frac{xy + yz + zx}{xyz}, \qquad \text{for all } x, y, z > 0$$

He begins an experiment, starting with a bottle of strength $a_0 = 1$. Each day he performs the same procedure: first blend the current bottle with a unit bottle (strength 1), then blend the result again with another unit bottle. If the strength after the n-th day is a_n, then

$$a_{n+1} = f(f(a_n, 1), 1), \qquad n \geq 0.$$

Determine $\left\lfloor \lim_{n\to\infty} a_n \right\rfloor$. Here $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x.

4. At time 0, Jerry is at $(1, 0)$ and Tom is at $(-1, 0)$. Jerry starts walking counterclockwise around the unit circle, and Tom starts creeping to the right along the x-axis. It so happens that Jerry's horizontal speed is always half of Tom's. If the shortest distance ever between the two is $\sqrt{m}/4$, where m is a natural number, report m as your answer.

Subjective Type Questions $(12 \times 5 = 60)$

*Each question carries **12 points**. Attempt as many as you can. Partially correct solutions will be rewarded accordingly.*

1. Wes and Paul play the following game. In the equation $x^3 + ax^2 + bx + c = 0$, starting with Wes, the players alternately choose one of the coefficients a, b, c which has not been chosen before, and replace it with a real number. Wes wins if the resulting equation has 3 distinct real zeros. Paul wins if Wes doesn't. Determine which of the two players has a winning strategy. If neither of them has a winning strategy, prove it.

2. Suppose that $f : [a, b] \to (0, \infty)$ and $g : [a, b] \to \mathbb{R}$ are continuous such that $g(a) = 0$ and $\int_a^b g(x)dx = 0$. Show that there is some $c \in (a, b)$ that satisfies
$$g(c) \int_a^c f(x)dx = f(c) \int_a^c g(x)dx.$$

3. In a chess tournament with $n \geq 5$ players, each player played all other players. One gets a point for a win, half a point for a draw, and zero points for a loss. At the end of the tournament, each player had a different number of points. Prove that the second and the third ranked players had together more points than the winner of the tournament.

4. Imagine that the films made by Alfred, Bergman, Coppola and Dario are assigned distinct positive integer scores a, b, c and d, respectively, representing their films' ratings by a panel of critics. Surprisingly, it is found that after raising their scores to some prime number p, the transformed ratings satisfy
$$a^p + b^p = c^p + d^p.$$
Show that if we compare Alfred with Coppola, and Bergman with Dario, then the sum of the difference in their original ratings must satisfy
$$|a - c| + |b - d| \geq p.$$
(Hint: You may find it interesting to know that Coppola likes calculus while Bergman likes number theory.)

5. Let ΔABC be a triangle with circumcentre O. Let P be a point on AB such that $\angle BOP = \angle ABC$ and Q be a point on AC such that $\angle COQ = \angle ACB$. Draw a perpendicular on PQ from O. Let it intersect the circumcircle of ΔAPQ at E. Prove that $\square AECB$ is a trapezium. Extend OE to intersect BC at F. Prove that O is the orthocentre of ΔFPQ.

End of Paper