Full Marks: 100 Category: Sub-junior Time: 3 hours

Section A: Objective Questions (MCQs)

*Each question carries **4 points**. There are **10 questions** in **Section A**. Briefly **justify** your answers. A correct answer without proper justification might **not** fetch complete **credit**.*

1. If $x^x = 2^{64}$, what is the value of x?

   (A) 8 **(B)** 16 **(C)** 32 **(D)** 64

2. A student has a collection of 5-rupee and 10-rupee coins with a total value of Rs. 140. If the number of 5-rupee coins is swapped with the number of 10-rupee coins, the total value of the collection becomes Rs. 160. What is the total number of coins the student has?

   (A) 15 **(B)** 18 **(C)** 20 **(D)** 25

3. If real numbers a and b satisfy $a+b = 5$ and $a^2+b^2 = 17$, what is the value of a^3+b^3?

   (A) 55 **(B)** 60 **(C)** 65 **(D)** 70

4. You can fill each of the blanks using $+, -, \times$, or $\div$.

$$1 \,\square\, 2 \,\square\, 3 \,\square\, 4 \,\square\, 5 \,=\, 0$$

   What should the first operation from the left be, in order to make the result 0?

   (A) $+$ **(B)** $-$ **(C)** $\times$ **(D)** $\div$

5. The first three terms of a geometric progression are x, y, and z, where all terms are distinct and non-zero. If the sequence $x, 2y, 3z$ forms an arithmetic progression, what is the common ratio of the geometric progression?

   (A) $\dfrac{1}{2}$ **(B)** $\dfrac{1}{3}$ **(C)** $\dfrac{2}{3}$ **(D)** 3

6. Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$ be a polynomial with real coefficients. If it is given that $P(1) = 1$, $P(2) = 2$, $P(3) = 3$, and $P(4) = 4$, find the value of $P(5)$.

   (A) 5 **(B)** 24 **(C)** 29 **(D)** 125

7. How many 3-digit positive integers have the property that the sum of their digits is exactly equal to 6?

   (A) 15 **(B)** 18 **(C)** 21 **(D)** 28

8. A star is drawn by extending the sides of a regular pentagon as shown in the figure. What proportion of the (larger) star given below is shaded?
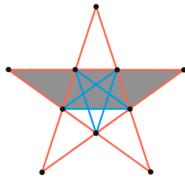
(A) $\dfrac{3}{8}$ (B) $\dfrac{1}{2}$ (C) $\dfrac{5}{8}$ (D) $\dfrac{2}{3}$

9. A certain principal amount is invested at a fixed rate of compound interest, compounded annually. The interest earned exactly during the second year is Rs. 1200, and the interest earned exactly during the third year is Rs. 1440. What was the original principal amount invested?

(A) Rs. 4000   (B) Rs. 5000   (C) Rs. 6000   (D) Rs. 7200

10. Two metallic alloys contain gold and copper. In the first alloy, the ratio of gold to copper is 2 : 3, and in the second alloy, the ratio is 3 : 7. If equal weights of both alloys are melted together to form a completely new alloy, what is the percentage of gold in the resulting mixture?

(A) 30%   (B) 35%   (C) 40%   (D) 45%

Section B: Subjective Problems

*Each question carries **10 points**. There are **6 questions** in **Section B**. Attempt **ALL** questions. Partially correct answers will be rewarded accordingly.*

1. Let $P := \{p_1, p_2, p_3, \dots\}$ denote the set of prime numbers written in ascending order, where p_k denotes the k-th prime number.

   (a) Explicitly state the value of p_1 and p_{10}.

   (b) Find the smallest $k \in \mathbb{N}$ such that $p_{k+2} - p_{k+1} = p_{k+1} - p_k = 2$.

   (c) Find the smallest $k \in \mathbb{N}$ for which there exists $l \in \mathbb{N}$ such that $p_k = \dfrac{p_l + p_{l+1}}{2}$

   (d) Find the smallest $k \in \mathbb{N}$ for which there exists $l \in \mathbb{N}$ such that $p_k = \sqrt{p_l \cdot p_{l+1}}$

   $$(1+2+3+4=10)$$

2. (a) Find all pairs (x, y) such that

   $$2x + 3y = 6$$
   $$3x + 4.5y = 9$$

   (b) Find all pairs (x, y) such that

   $$3x + 5y = 25$$
   $$6x + 10y = 45$$

   $$(5+5=10)$$

3. **Akash** has decided to surprise his friend **Shreja** with a basket of rare *glowing magical lotuses* that grow near the Enchanted Forest. Unfortunately, the path to **Shreja**'s house is not simple.

   To reach her, **Akash** must cross the mysterious **River of Reflections** exactly three times. The river has a strange magical property: *every time **Akash** crosses it, the number of lotuses he is carrying instantly doubles.*

   However, some rather troublesome characters lie in wait along the way.

(a) After crossing the river for the first time, **Akash** is stopped by **Pat**, a flashy traveller who insists that visitors must give him some lotuses before continuing.

(b) After crossing the river for the second time, **Akash** encounters **Misty**, a very large and rather slow-witted forest monster. **Misty** does not think too much, but **he** does know one rule: **he** must receive *exactly the same number of lotuses* that **Pat** received.

(c) After crossing the river for the third and final time, **Akash** finally reaches **Shreja** and gives *all the lotuses he has left* to her.

Akash wants everything to be perfectly fair. He plans the journey so that:

(a) **Pat**, **Misty**, and **Shreja** each receive exactly the same number of lotuses.

(b) After giving the lotuses to **Shreja**, **Akash** has 0 lotuses left.

(c) **Akash** can only pluck whole lotuses (no fractions allowed).

What is the *minimum number of lotuses* **Akash** must pluck at the beginning before crossing the river for the first time?

How *many* lotuses does **Shreja** receive? (7+3=10)

4. A, B, C, D and E live in an island called MTRPia. The following facts are known to us:

  (i) Each of these five people is either a human or a werewolf.

 (ii) Humans always tell the truth and werewolves always lie.

Each of them has made a statement about them, which is presented in the table below:

Person	Statement
A	B and E are either both humans or both werewolves.
B	C and E are either both humans or both werewolves.
C	Exactly two of us are humans.
D	A and B are either both humans or both werewolves.
E	C is a werewolf.

Among these five persons, how many are werewolves? Justify your answer.

5. The figure below shows a regular heptagon (shaded) with its sides extended to form a star. Find the value of x. (10)
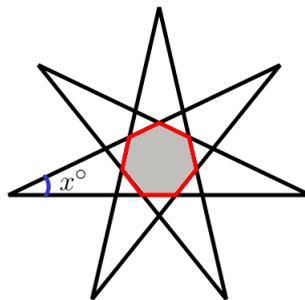
6. A natural number is *good* if it can be expressed as the sum of the squares of two natural numbers (for example, $2 = 1^2 + 1^2$, $85 = 9^2 + 2^2$).

(a) Show that the product of two *good* numbers is also *good*.

    Hint: Can you write $(a^2 + b^2)(c^2 + d^2)$ in terms of $(ac + bd)$ and $(ad - bc)$?

(b) Find the smallest *good* number that can be expressed as the sum of two squares in two different ways.

(c) Using part (a) or otherwise, find a *good* number that can be expressed as the sum of two squares in at least 3 different ways.