Mathematics Talent Reward Programme
Question Paper for Junior Category

2015

Total Marks: 150 Allotted Time: 2:00 p.m. to 4:30 p.m.

Multiple Choice questions

[You should answer these questions in the first page according to the order given in the question. Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. In a ceremony all the guests present shook hands with each other. There were 55 handshakes observed. How many guests were there?
A. 9 B. 10 C. 11 D. 12

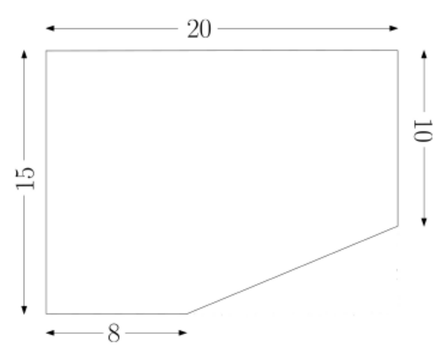
2. Let x be a two-digit prime number such that if its digits are interchanged, we get a new prime number y. If the difference between x and y is 18 , then what is the value of $5xy$?
A. 2005 B. 2015 C. 2025 D. 2035

3. How many times during a day do the hour hand and the minute hand of a clock make a right angle?
A. 23 B. 24 C. 25 D. None of these.

4. Two trains start from A and B and travel towards each other at 50 km/h and 60 km/h respectively. At the time of meeting the second train has travelled 120 km more than the first one. Find the initial distance between them in kilometers.
A. 1280 B. 1320 C. 1300 D. 1380

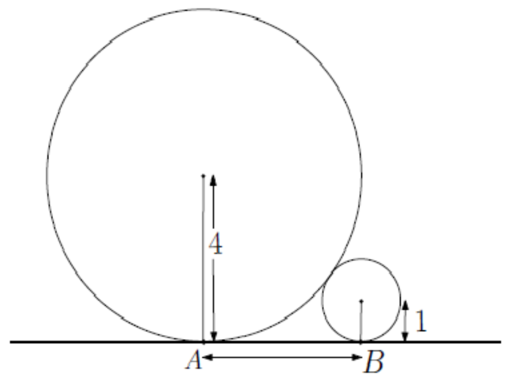
5. What is the last digit of 7^{2015} ?
A. 1 B. 3 C. 7 D. 9

6. What is the area of the region in the figure below?



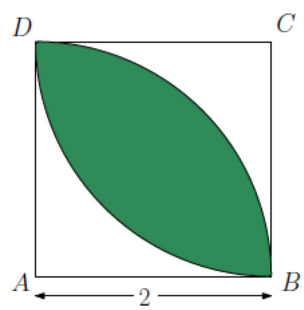
a) 250 b) 270 c) 300 d) 320

7. What is the length of the segment AB in the figure below?



a) 4 b) 5 c) 3 d) $3\sqrt{3}$.

8. If the figure below, the square has sides of length 2 units. Two arcs of a circle are drawn with centers at A and C respectively and radius 2 units. What is the area of the shaded region?



a) 2π b) $2\pi - 3$ c) $2(\pi - 2)$ d) $2(\pi - 3)$

9. A can contains a mixture of two liquids A and B in the ratio 7:5. When 9 litres of the mixture are drawn and replaced by the same amount of liquid B, the ratio of A and B becomes 7:9. How many litres of liquid A was contained in the can initially?

a) 18 b) 19 c) 20 d) None of these

10. From a square with sides of length 2 m, corners are cut away so as to form a regular octagon. What is the area of the octagon in sq.m?

a) $2\sqrt{3}$ b) $\frac{4}{\sqrt{3}}$ c) $4(\sqrt{2} - 1)$ d) None of these

11. Solve for x and y : $\sqrt{9x^2 - 30x + 74} + \sqrt{4y^2 + 28y + 74} = 12$

a) $x = \frac{5}{3}, y = \frac{7}{2}$ b) $x = \frac{5}{3}, y = -\frac{7}{2}$ c) $x = \frac{7}{2}, y = \frac{5}{3}$ d) $x = \frac{7}{2}, y = -\frac{5}{3}$

12. Define a sequence by $a_0 = a_1 = 1$ and $a_n = a_{n-1}a_{n-2} + 1$ for $n > 1$. Then

a) a_{2015} is odd and a_{2016} is odd
b) a_{2015} is even and a_{2016} is odd
c) a_{2015} is odd and a_{2016} is even
d) a_{2015} is even and a_{2016} is even

13. Consider the point $(4, 0)$ on the x, y-plane. Now if axes are rotated $45°$ anticlockwise then what are the new co-ordinates of this point in the new system?

    a) $(2\sqrt{2}, 2\sqrt{2})$ b) $(2\sqrt{2}, 0)$ c) $(2\sqrt{2}, -2\sqrt{2})$ d) $(4\sqrt{2}, 2\sqrt{2})$.

14. Find the area of the region in the co-ordinate plane which satisfies all the following inequalities.

$$\begin{aligned} x + y &\leq 1 \\ x - y &\leq 1 \\ y - x &\leq 1 \\ x + y &\geq -1 \end{aligned}$$

    a) $\sqrt{2}$ b) 2 c) $2\sqrt{2}$ d) 4

15. Consider a hollow paper cone with slant height 4 cm. It is cut along the slant surface and unfolded to make a sector. This sector subtends an angle of $60°$ at the center. What is the surface area of the cone in sq.cm?

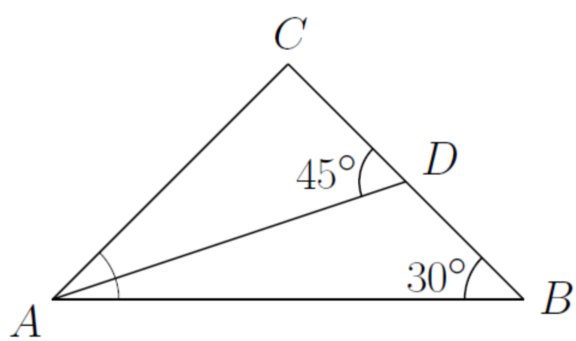
    a) $\frac{5\pi}{3}$ b) 2π c) $\frac{8\pi}{3}$ d) 3π

Short Answer Type Questions

[Each question carries a total of 15 marks. Credit will be given to partially correct answers]

1. A 15 cm long column of ants starts crawling. A rebel ant at the end of the column steps out and starts marching forward at a higher speed than the column. On reaching the front of the column, it immediately turns around and marches back at the same speed. When he reaches the end of the column he finds that the column of the remaining ants has moved exactly 15 cm. What distance did the rebel ant travel?

2. Consider the numbers $1, 2, 3, 4, \ldots, 13$. Can these numbers be partitioned into two groups such that the products of the elements are same in both groups?



3. In triangle $\triangle BAC$ with $\angle ABC = 30°$.D is the midpoint of BC. We join A and D and $\angle CDA = 45°$. Find $\angle BAC$

4. Let P, Q, R, S be the midpoints of the sides AB, BC, CD and DA respectively of a rectangle $ABCD$. If the area of the rectangle is Δ, then calculate the area bounded by the straight lines AQ, BR, CS and DP in terms of Δ.

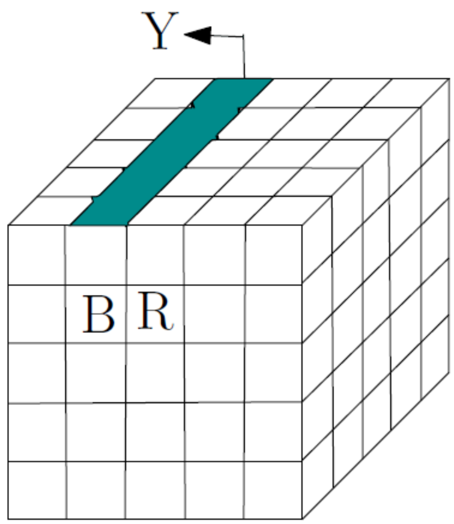
5. A pentagon is inscribed inside a fixed circle. Show that for the area of the pentagon to be maximum, it must be a regular one.

6. There are 125 unit cubes each of whose faces are coloured with blue, green and red such that each colour is used at least once and opposite faces have the same colour. Now, a $5 \times 5 \times 5$ cube is constructed using these small cubes such that the touching faces are of the same colour.

   i) Suppose two adjacent squares on one face of that whole cube are of different colours (say blue and red as in the figure below). Then show that the entire column containing the red square will be coloured red. Also the same should hold for the column containing the blue square.

   ii) Show that the squares in the strip marked Y should also be of the same colour. Also find this colour. [3]

   iii) Hence or otherwise show that there exists a face of the large cube in which all the squares are of the same colour.



Use of calculators is not allowed. You may use a ruler and a compass for construction.
~ Best of Luck ~

Mathematics Talent Reward Programme

Model Solutions for Junior Category

Multiple Choice Questions

[Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. (C) [Observe n people will have $\binom{n}{2}$ handshakes]
2. (B) [$x = 10a + b \Rightarrow |9a - 9b| = 18 \Rightarrow |a - b| = 2$, now check all possibilities]
3. (D) [minute hand completes 11 rotations wrt hour hand, 2 right angles each rotations]
4. (B) [2^{nd} train travels 10km more in each hour, hence time of journey is 12 hour, and their relative speed is 110 km/h]
5. (B) [$7^1 \equiv 7, 7^2 \equiv 9, 7^3 \equiv 3$, $7^4 \equiv 1$ (mod 10). Now observe cyclicity]
6. (B) [Complete the rectangle, find the area, then subtract the area of added right angled triangle]
7. (A) [Observe that distance between centres 5, then complete the right angle and apply pythagorean theorem]
8. (C) [$2 \times area(arc(ABD) - \Delta ABD)$]
9. (D) [$7x : 5x + 9 = 7 : 9$, find $7x + \frac{9 \times 7}{7+5}$]
10. (D)[If the side of the octagon is x, then looking at the right angled triangle in the corner, we get $2(1 - \frac{x}{2})^2 = x^2$, solving x and subtracting the areas of those triangles from the square, we get the result]
11. (B)[Observe $9x^2 - 30x + 74 \geq 49$ and $4y^2 + 28y + 74 \geq 25$]
12. (C) [Observe $\{a_i\}_{n=0}^{\infty} \equiv \{1, 1, 0, 1, 1, 0 \ldots\} (mod 2)$]
13. (C) [Find the point on x+y=0 which is 4 units away from origin]
14. (B) [Observe that the region is a square with diagonal 2]
15. (C) [Observe $\pi r l = \frac{1}{6}\pi l^2$]

Short Answer Type Questions

[Each question carries a total of 15 marks. Credit will be given to partially correct answers]

1. Let velocity of the column be u cm/s and velocity of the rebel ant be v cm/s. So, the relative velocity of the rebel ant with respect to the column will be $(v - u)$cm/s when going forward and $v + u$ cm/s when coming back.

   $\therefore$ Total time taken $= \frac{15}{v-u} + \frac{15}{v+u} s$. Also distance travelled by the column is $\left(\frac{15}{v-u} + \frac{15}{v+u}\right) \cdot u$ which is given to be 15 cm.

$$\begin{aligned}
&\therefore \left(\frac{15}{v-u} + \frac{15}{v+u}\right) \cdot u = 15 \\
\Longrightarrow &\frac{1}{\frac{v}{u} - 1} + \frac{1}{\frac{v}{u} + 1} = 1 \\
\Longrightarrow &2\left(\frac{v}{u}\right) = \left(\frac{v}{u}\right)^2 - 1 \\
\Longrightarrow &\left(\frac{v}{u}\right)^2 - 2\left(\frac{v}{u}\right) - 1 = 0 \\
\Longrightarrow &\left(\left(\frac{v}{u}\right) - 1\right)^2 = 2 \\
\Longrightarrow &\left(\frac{v}{u}\right) - 1 = \pm\sqrt{2} \\
\Longrightarrow &\left(\frac{v}{u}\right) = 1 + \sqrt{2} \left[\because 1 - \sqrt{2} < 0 \text{ and } \left(\frac{v}{u}\right) > 0\right]
\end{aligned}$$

   So, the distance travelled by the rebel ant is

$$\left(\frac{15}{v-u} + \frac{15}{v+u}\right) \cdot v = \left(\left(\frac{15}{v-u} + \frac{15}{v+u}\right) \cdot u\right) \cdot \frac{v}{u} = 15 \cdot \frac{v}{u} = 15(1 + \sqrt{2})\text{cm}$$

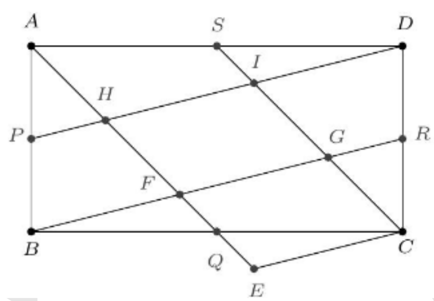
2. If we partition the numbers $1, 2, \ldots, 13$ into 2 groups then one of them must contain the number 13 and the other group will not. Then product pf the elements in the group containing 13 will be divisible by 13 whereas the product of the elements in he other group will not be divisible by 13. Hence a partition such that the product of elements in both groups is the same is not possible.

3. Consider a 3×3 chessboard and the label the squares as shown below.

1	4	7
2	5	8
3	6	9

Now consider the cyclic path of the knight's move $1 \to 8 \to 3 \to 4 \to 9 \to 2 \to 7 \to 6 \to 1$ So, in a 3×3 board the knight can move from any square to any square except the middlemost. Call this 3×3 board without the middlemost square a 3×3 ring. Now you can cover a 8×8 chessboard with 3×3 overlapping rings and so you can move from any square in a ring to any other square in that ring as well as the squares in rings with which it overlaps. Thus you can traverse the whole chessboard with a knight.

4. Draw a line through C parallel to BR. Let it intersect extended AQ at E. Let BR intersect AQ at F and SC at G and DP intersect them at H and I respectively. Now consider $\triangle BFQ$ and $\triangle CEQ$.



$$\begin{aligned} BQ &= QC \\ \angle BQF &= \angle CQE [\text{ Vertically Opposite Angle }] \\ \angle QBF &= \angle QCE [\because BF \| EC] \\ \therefore \triangle BFQ &\cong \triangle CEQ \end{aligned}$$

Now observe that in $\triangle ABQ$ and $\triangle CDS$

$$\begin{aligned} AB &= CD \\ BQ &= DS \\ \angle ABQ &= \angle CDS \\ \therefore \triangle ABQ &\cong \Delta CDS \\ \therefore \angle AQB &= \angle DSC = \angle SCQ [\because BC \| AD] \\ \implies AE &\| CS \end{aligned}$$

Similarly, we can show that $BR \| DP$.

Now we get that, $DP \| BR \| CE$ so, we can conclude that

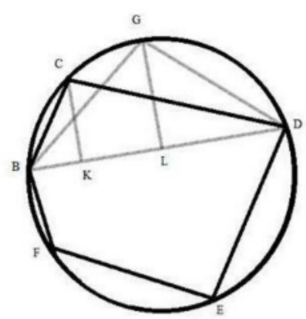
$$\frac{CG}{GI} = \frac{CR}{RD} = 1 \Longrightarrow CG = GI$$

Hence parallelograms $ECGF$ and $FGIH$ has same height [$\because AE \| CS$] and same base length [$\because CG = GI$]. So, they have same area. Hence,

$$\text{Area of } \triangle CDI = \text{ Area of } \triangle DAH = \text{ Area of } \triangle ABF$$

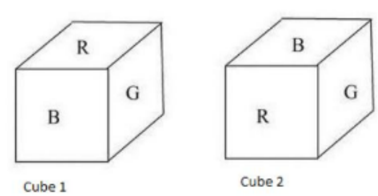
So finally we get Area of $FGIH = \frac{\text{Area of } ABCD}{5} = \frac{\Delta}{5}$.

5. Let us consider a pentagon $BCDEF$ inscribed in the circle which is irregular. Then $BCDEF$ has at least a pair of consecutive sides whose lengths are different. Let these sides be BC and CD (as shown in the figure). Now join BD and let G be the midpoint of the circular arc BCD. Perpendiculars CK and GL are dropped upon BD. It can be easily seen that $GL > CK$ and hence area of $\triangle BGD$ is greater than that of $\triangle BCD$ [as they both have same base BD]. Now area of the pentagon $BGDEF =$ area of the quadrilateral $BDEF+$ area of triangle BGD. Also area of the pentagon $BCDEF =$ area of the quadrilateral $BDEF+$ area of triangle BCD. Hence area of pentagon $BGDEF$ (which is also inscribed in the circle) is greater than the area of the pentagon $BCDEF$. So we conclude that $BCDEF$ cannot be a pentagon which has the maximum area among all the inscribed pentagons.

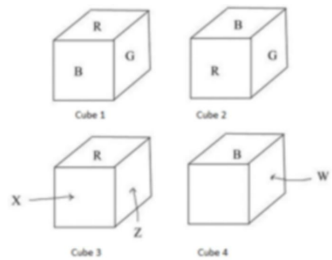


But note that if the pentagon was regular then the points C and G would have coincided, in other words, we could not have drawn another pentagon inscribed in the circle but with greater area. Thus for the area to be maximum the pentagon has to be regular.

6. (a) First we see that due to the colouring scheme each small cube has two blue faces, two red faces and two green faces. Let us draw a clearer picture of the adjacent cubes coloured red and blue.



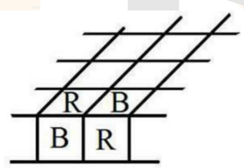
We see that the touching faces cannot be blue because then blue would be used in four of the faces. Similarly it cannot be red and hence must be green. Then the top face of cube 1 must be red and that of cube 2 must be blue. Now we will look at the cube just below cube 1 and cube 2 . Let us call them cube 3 and cube 4 respectively.



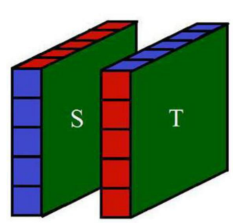
As the bottom face of cube 1 is also red the top face of cube 3 must be red. (As touching faces are of the same colour). Similarly the top face of cube 4 is blue. Now the face marked X is either blue or green. If it is green then face marked Z is blue and the face of cube 4 touching Z must also be blue. But the top face of cube 4 is blue which makes blue appear 4 times in cube 4 . So, X must be blue and Z is green. Hence W is also green which leaves the front face of cube 4 red in colour.

In the same process the cube lying below cube 3 will have blue colour in its front face and so on for the all the cubes lying below. By a similar argument the cubes lying above cube 1 will have blue colour in their front faces. So, the entire column will be blue in colour. Similarly the entire column containing the face with red colour will be red.

(b) Now consider the cubes lying at the top of the blue column and red column. It can easily be seen that their top faces are red and blue respectively as the touching face will be green as shown before.



Now rotate the large cube so that the face lying at the top previously now faces you. Then by the argument of part (a) the entire column containing the square with the red face must be coloured red. As P is a part of the strip Y, the colour of Y is red.
(c) Assume that none of the faces of the larger cube has squares of the same colour. Then there exists a face where there are adjacent squares of different colour. Let us assume that they are red and blue. By the argument in part (a) and (b) of the problem we get a figure as shown below.



Then the surfaces marked S and T will be green. As all touching faces and opposite faces are of the same colour the face of the large cube lying parallel to S and T will be green. Similarly if we would have started with the adjacent squares being red and green then we would have gotten a blue coloured face. In the remaining case of them being blue and green we would get a red face. So in any case we get a face with all 25 squares in it of the same colour.

Mathematics Talent Reward Programme

Question Paper for Senior Category

18^{th} January,2015

Total Marks: 150 Allotted Time: 10:00 a.m. to 12:30 p.m.

Multiple Choice Questions

[You should answer these questions in the first page according to the order given in the question. Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. How many distinct arrangements are possible for wearing five different rings in the five fingers of the right hand? (We can wear multiple rings in one finger)

   a) $\frac{10!}{5!}$, b) 5^5, c) $\frac{9!}{4!}$, d) None of these.

2. Let $f_n(x) = \underbrace{xx\cdots x}_{n \text{ times}}$ that is, $f_n(x)$ is an n digit number with all digits x, where $x \in \{1, 2, \ldots, 9\}$. Then which of the following is $(f_n(3))^2 + f_n(2)$?

   a) $f_n(5)$ b) $f_{2n}(9)$, c) $f_{2n}(1)$, d) None of these.

3. If $A_i = \frac{x-a_i}{|x-a_i|}, i = 1, 2, \ldots n$ for n numbers $a_1 < a_2 < \cdots < a_m < \cdots < a_n$, then $\lim_{x\to a_m}(A_1A_2\ldots A_n) =$?

   a) $(-1)^{m-1}$, b) $(-1)^m$, c) 1 , d) Does not exist.

4. Let n be an odd integer. Placing no more than one **X** in each cell of a $n \times n$ grid, what is the greatest number of **X** 's that can be put on the grid without getting n**X** 's together vertically, horizontally or diagonally?

   a) $2\binom{n}{2}$, b) $\binom{n}{2}$, c) $2n$, d) $2\binom{n}{2} - 1$.

5. How many integral solutions are there for the equation $x^5 - 31x + 2015 = 0$?

   a) 2 b) 4 c) 1 d) None of these.

6. Let AC and CE be perpendicular line segments, each of length 18 . Suppose B and D are the midpoints of AC and CE respectively. If F be the point of intersection of EB and AD, then the area of $\triangle BDF$ is?

   a) $27\sqrt{2}$, b) $18\sqrt{2}$, c) 13.5 , d) 18

7. How many x are there such that $x, [x], \{x\}$ are in harmonic progression (i.e, the reciprocals are in arithmetic progression)? (Here $[x]$ is the largest integer less than equal to x and $\{x\} = x - [x]$)

   a) 0 b) 1 c) 2 d) 3

8. In $\triangle ABC, AB = AC$ and D is foot of the perpendicular from C to AB and E the foot of the perpendicular from B to AC, then

   a) $BC^3 > BD^3 + BE^3$, b) $BC^3 < BD^3 + BE^3$,

   c) $BC^3 = BD^3 + BE^3$, d) None of these.

9. How many 5×5 grids are possible such that each element is either 1 or 0 and each row sum and column sum is 4 ?

   a) 64 b) 32 c) 120 d) 96

10. If $\sum_{i=1}^{n} \cos^{-1}(\alpha_i) = 0$, then find $\sum_{i=1}^{n} \alpha_i$.

    a) $\frac{n}{2}$, b) n c) nπ d) $\frac{n\pi}{2}$.

11. $S = \{1, 2, \ldots, 6\}$. Then find out the number of unordered pairs of (A, B) such that $A, B \subseteq S$ and $A \cap B = \emptyset$.

    a) 360 b) 364 c) 365 d) 366

12. The maximum value of $\sin^4\theta + \cos^6\theta$ will be?

    a) $\frac{1}{2\sqrt{2}}$, b) $\frac{1}{2}$, c) $\frac{1}{\sqrt{2}}$, d) 1.

13. Define $f(x) = \max\{\sin x, \cos x\}$. Find at how many points in $(-2\pi, 2\pi)$, $f(x)$ is not differentiable?

    a) 0 b) 2 c) 4 d) ∞

14. $z = x + iy$ where x and y are two real numbers. Find the locus of the point (x, y) in the plane, for which $\frac{z+i}{z-i}$ is purely imaginary (that is, it is of the form ib where b is a real number). [Here, $i = \sqrt{-1}$]

    a) A straight line b) A circle c) A parabola d) None of these

15. Find out the number of real solutions of $x^2 e^{\sin x} = 1$

    a) 0 b) 1 c) 2 d) 3

Short Answer Type Questions

[Each question carries a total of 15 marks. Credits will be given to partially correct answers]

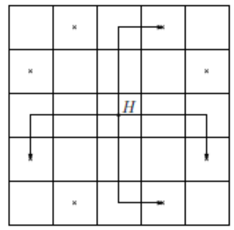
1. In a room there is a series of bulbs on a wall and corresponding switches on the opposite wall. If you put on the n-th switch the n-th bulb will light up. There is a group of men who are operating the switches according to the following rule: they go in one by one and starts flipping the switches starting from the first switch until he has to turn on a bulb; as soon as he turns a bulb on, he leaves the room. For example the first person goes in, turns the first switch on and leaves. Then the second man goes in, seeing that the first switch on turns it off and then lights the second bulb. Then the third person goes in, finds the first switch off and turns it on and leaves the room. Then the fourth person enters and switches off the first and second bulbs and switches on the third. The process continues in this way. Finally we find out that first 10 bulbs are off and the 11-th bulb is on. Then how many people were involved in the entire process?

2. Let x, y be numbers in the interval $(0, 1)$ such that for some $a > 0, a \neq 1$

$$\log_x a + \log_y a = 4\log_{xy} a.$$

   Prove that $x = y$.

3. Let $n > 3$ be an integer. Show that, in an $n \times n$ chessboard, it is possible to traverse to any given square from another given square using a knight. (A knight can move in a chessboard by going two steps in one direction and one step in a perpendicular direction as shown in the given figure.)

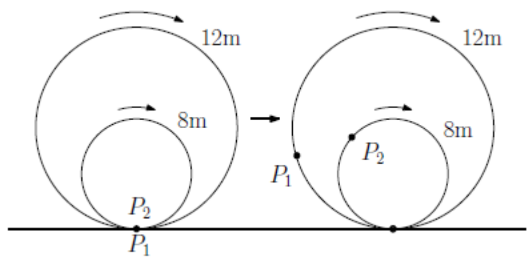


4. Find all real numbers $x_1, x_2, \ldots, x_n$ satisfying,

$$\sqrt{x_1 - 1^2} + 2\sqrt{x_2 - 2^2} + \cdots + n\sqrt{x_n - n^2} = \frac{1}{2}(x_1 + x_2 + \cdots + x_n)$$

5. Let a be the smallest and A the largest of n distinct positive integers. Prove that the least common multiple of these numbers is greater than or equal to na and that the greatest common divisor is less than or equal to $\frac{A}{n}$.

6. In the following figure, the bigger wheel has circumference 12 m and the inscribed wheel has circumference 8 m. P_1 denotes a point on the bigger wheel and P_2 denotes a point on the smaller wheel. Initially P_1 and P_2 coincide as in the figure. Now we roll the wheels on a smooth surface and the smaller wheel also rolls in the bigger wheel smoothly. What distance does the bigger wheel have to roll so that the points will be together again?



Use of calculators is not allowed. You may use a ruler and a compass for construction.

~ Best of Luck ~

Mathematics Talent Reward Programme

Model Solutions for Senior Category

Multiple Choice Questions

[Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. (C) [Arrange rings in 10! ways, then find the number of nonnegative integer solutions of $a+b+c+d+e=10$]
2. (C) [Observe $f_n(k) = kf_n(1)$, hence $(f_n(3))^2 + f_n(2) = (9f_n(1)+2)f_n(1) = f_{2n}(1)$]
3. (D)[right hand limit and left hand limit have opposite signs]
4. (A)[Placing X everywhere except a diagonal is a valid construction, i.e. n(n-1) X's. Also, putting more than n(n-1) X's makes atleast one row full by PHP]
5. (D) [0 solutions]
6. (C) [triangle formed by centroid and two vertices has $\frac{1}{3}$rd area of actual triangle, and median divides area by half]
7. (B) [$\frac{1}{x} + \frac{1}{\{x\}} = \frac{2}{[x]}$, then put $\{x\} = x - [x]$, get $[x] = \sqrt{2}(x-[x])$)]
8. (A) [Observe $BC^2 = BD^2 + BE^2$,$BC > BD, BC > BE$,hence $BC^3 = (BD^2+BE^2)(BC) > BD^3 + BE^3$]
9. (C) [We have 6-i choices of placing the 0 in each row]
10. (B) [$cos^{-1}(x) \geq 0$, hence $a_i = \frac{\pi}{2} \forall i$]
11. (C) [Chose k elements for A, chose B in 2^{6-k} ways, sum and divide by 2. Then add the case (ϕ, ϕ)]
12. (D) [Check the maxima]
13. (C) [sinx and cosx graph intersect each other at 4 points]
14. (B) [$\bar{a} = -a$ when a is imaginary]
15. (C) [$f(x) = x^2 e^{sinx}$, then f(0)=0, f($\frac{5\pi}{2}$)=f($\frac{-5\pi}{2}$)=$\frac{25\pi^2}{4} > 1$, hence atleast two solutions, and observe there are exactly two solutions]

Short Answer Type Questions

[Each question carries a total of 15 marks. Credit will be given to partially correct answers]

1. When the n-th person leaves the room, the bulbs can be represented as a binary number, say A_n whose k-th digit from right is 1 if k-th bulb is ON and 0 if k-th bulb is OFF. Denote by $(B)_{10}$ to be the decimal representation of a binary number B. Then we will show that $(A_n)_{10} = n$.

   Clearly when the 1 st person leaves, only the first switch is turned on. Therefore $A_1 = 1 \implies (A_1)_{10} = 1$. Now we use induction. Suppose it $(A_n)_{10} = n$ holds for all $n \leq m$. We will show $(A_{m+1})_{10} = m+1$. If the 1st switch is turned off when the $(m+1)$-th person enters, then $(A_{m+1})_{10} - (A_m)_{10} = 1 \implies (A_{m+1})_{10} = m+1$. If the first r switches are turned on and $(r+1)$-th switch is turned off when the $(m+1)$-th person enters, then

   $$(A_{m+1})_{10} - (A_m)_{10} = (\dots 1\underbrace{00\dots0}_{r \text{ times}})_{10} - (\dots 0\underbrace{11\dots1}_{r \text{ times}})_{10} = 2^r - \sum_{k=0}^{r-1} 2^k = 1$$

   Hence by induction the claim is true. We use this method to compute the number of involved people. Now note that our configuration gives A_n to be a binary number with 1 followed by 10 0 's. So, n, the number of persons is

   $$(\underbrace{100\dots0}_{10 \text{ times}})_{10} = 2^{10} = 1024$$

2. From the given equation we have

$$
\begin{aligned}
&\log_x a + \log_y a = 4\log_{xy} a\\
\Longrightarrow &\frac{\log a}{\log x} + \frac{\log a}{\log y} = 4\frac{\log a}{\log xy}\\
\Longrightarrow &\log a\left(\frac{1}{\log x} + \frac{1}{\log y}\right) = \frac{4\log a}{\log x + \log y}
\end{aligned}
$$

Since $a > 0$ and $a \neq 1$, this ensures $\log a \neq 0$. Hence we can cancel $\log a$ both sides to get

$$
\begin{aligned}
&\frac{1}{\log x} + \frac{1}{\log y} = \frac{4}{\log x + \log y}\\
\Longrightarrow &2 + \frac{\log y}{\log x} + \frac{\log x}{\log y} = 4\\
\Longrightarrow &\left(\sqrt{\log_x y} - \sqrt{\log_y x}\right)^2 = 0
\end{aligned}
$$

Now note that if $t \in (0,1)$, then $\log t < 0$. Therefore as $x, y \in (0,1), \log_y x = \frac{\log x}{\log y} > 0$. Thus taking square roots is justified. Therefore

$$
\log_x y - \log_y x = 0 \Longrightarrow (\log x)^2 - (\log y)^2 = 0 \Longrightarrow \log x - \log y = 0 \Longrightarrow x = y
$$

Since $\log x, \log y$ are both negative $\log x + \log y = 0$ was ruled out.

3. We prove by inducting on n. Label the unit squares as (x, y) where $x, y \in \{1, 2, \ldots, n\}$ and the top left corner is $(1,1)$. Consider the graph G_n with the n^2 unit squares as its vertices and two vertices are joined if and only if we can go form one point to the other via just one knight move. We want to prove that this G_n is connected.

   Base Case: $n = 4$. One can easily observe this by just looking at the graph for this case.

   Inductive Step. Suppose G_k is connected. One horse move gives us the following map for a vertex (x, y).

$$
\begin{aligned}
(x, y) &\mapsto \{(x+2, y+1), (x+2, y-1), (x-2, y+1), (x-2, y-1)\}\\
(x, y) &\mapsto \{(x+1, y+2), (x+1, y-2), (x-1, y+2), (x-1, y-2)\}.
\end{aligned}
$$

   Note that G_n is connected if and only we can go from any point to $(1,1)$. By induction hypothesis, one can go to $(1,1)$ from any vertex $(x, y) \in S = \{(p, q) \mid p, q \in \{1, \ldots, k\}\}$. So we just need to take care of vertices of the form $(k+1, t)$ and $(t, k+1)$.

   For a vertex of the form $(k+1, t)$ where $t \in \{1, \ldots, k+1\}$, consider the map

$$
(k+1, t) \mapsto (k, t \pm 2) \quad \text{depending on whether } t = 1, 2 \text{ or } k-1, k, k+1.
$$

   For a vertex of the form $(t, k+1)$ where $t \in \{1, \ldots, k\}$, consider the map

$$
(t, k+1) \mapsto (t \pm 2, k) \quad \text{depending on whether } t = 1, 2 \text{ or } k-1, k.
$$

   Now all resulting vertices are in S, hence there is a path to $(1,1)$. Hence G_{k+1} is connected.

4. From the given equation we have

$$
\begin{aligned}
&\sum_{k=1}^{n} k\sqrt{x_k - k^2} = \frac{1}{2}\sum_{k=1}^{n} x_k \\
\Longrightarrow &\sum_{k=1}^{n}\left(x_k - 2k\sqrt{x_k - k^2}\right) = 0 \\
\Longrightarrow &\sum_{k=1}^{n}\left(\sqrt{x_k - k^2} - k\right)^2 = 0 \\
\Longrightarrow & x_k - k^2 = k^2 \forall k = 1, \ldots, n \\
\Longrightarrow & x_k = 2k^2 \forall k = 1, \ldots, n
\end{aligned}
$$

5. Let $a = a_1 < a_2 < \cdots < a_n = A$ be the n distinct positive integers. Suppose d and l are the gcd and lcm of these numbers respectively. Then $\frac{a_i}{d}$ are all positive integers. Also

$$1 \leq \frac{a_1}{d} < \frac{a_2}{d} < \cdots < \frac{a_n}{d}$$

Since $\frac{a_i}{d}$ is an integer strictly greater than $\frac{a_{i-1}}{d}$, we have $\frac{a_i}{d} \geq 1 + \frac{a_{i-1}}{d}$. Therefore

$$\frac{a_n}{d} \geq 1 + \frac{a_{n-1}}{d} \geq 1 + 1 + \frac{a_{n-2}}{d} \geq \cdots \geq n - 1 + \frac{a_1}{d} \geq n \Longrightarrow d \leq \frac{a_n}{n} = \frac{A}{n}$$

Similarly, we have that $\frac{l}{a_i}$ are all integers and

$$1 \leq \frac{l}{a_n} < \frac{l}{a_{n-1}} < \cdots < \frac{l}{a_1}$$

Since $\frac{l}{a_i}$ is an integer strictly greater than $\frac{l}{a_{i+1}}$, we have $\frac{l}{a_i} \geq 1 + \frac{l}{a_{i+1}}$. Therefore

$$\frac{l}{a_1} \geq 1 + \frac{l}{a_2} \geq 1 + 1 + \frac{l}{a_3} \geq \cdots \geq n - 1 + \frac{l}{a_n} \geq n \Longrightarrow d \geq na_1 = na$$

6. First observe that the points P_1 and P_2 can meet together only when both of them are on the ground surface. Suppose the wheels starts initially at point A on the surface and they meet again at point B on the surface t m away. Suppose the bigger wheel revolves k times to reach B. Therefore $12k = t$. So, 12 divides t. Similarly 8 must also divide t. Thus, t must be divisible by 24 . Since the points meet for the first time after A at B, value of t must be the minimum possible multiple of 24 , that is the required distance is 24 m.

Mathematics Talent Reward Programme

Question Paper for Junior Category

17^{th} January,2016

Total Marks: 150 — Allotted Time: 2:00 p.m. to 4:30 p.m.

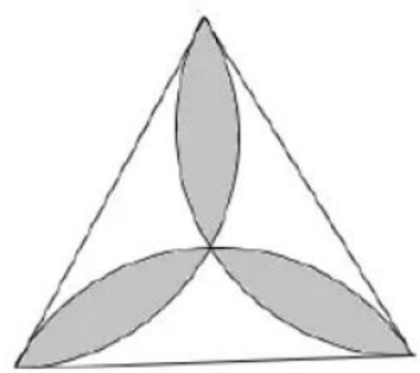
Multiple Choice Questions

[You should answer these questions in the first page according to the order given in the question. Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. Radius of a cone is increased by 10% and height of the same cone is decreased by 10%, then the volume of the cone has increased by

   a) 8.3%, b) 8.6%, c) 8.9%, d) 9.1%.

2. Consider an equilateral triangle of length $\sqrt{6}$ as shown in the figure. Find the area of the shaded portion.



   a) $\frac{3}{2}(2\pi - \sqrt{6})$, b) $2\pi - 3\sqrt{3}$, c) $\sqrt{6}\pi - \frac{1}{\sqrt{3}}$, d) $\frac{3}{2}(3\pi - 2\sqrt{3})$.

3. Perimeter of a triangle with sides a, b and c is 2 . Then the expression $ab + bc + ca - abc - 1$ is

   a) always positive b) always negative c) 0 d) None of these

4. x, y are real numbers with $x + y = 1$ and $x^2 + y^2 = 2$. Then the value of $x^5 + y^5$ is given by

   a) $\frac{5}{2}$, b) $\frac{17}{4}$, c) $\frac{7}{2}$, d) $\frac{3}{2}(3\pi - 2\sqrt{3})$.

5. A five digit number is called Flappy if product of its last two digits is 32 and sum of all five digits is 36 . Suppose

$$x = \frac{\text{Number of Flappy numbers}}{\text{Number of Flappy numbers which are divisible by 36}}$$

   Then x equals

   a) 1 b) 3 c) 2 d) none of these

6. You are given three bricks each measuring $5'' \times 4.5'' \times 3''$. How many different heights can you build up using all three of them?

   a) 14 b) 7 c) 10 d) 13

7. Let $x_1 = 2016$. For $n > 1$ define $x_n = \frac{n}{x_{n-1}}$. Then $x_1 x_2 \cdots x_{10} =$

a) 2016 b) 2280 c) 3684 d) None of these

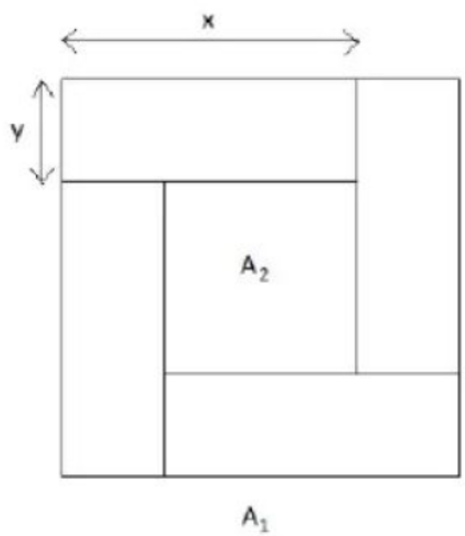
8. Vessel A has liquids X and Y in the ratio $X : Y = 8 : 7$. Vessel B holds a mixture of X and Y in the ratio $X : Y = 5 : 9$. What ratio should you mix the liquids in both vessels if you need the mixture to be $X : Y = 1 : 1$?

a) 4:1 b) 30:7 c) 17:25 d) 7:30

9. How many six digit perfect squares can be formed using all the numbers $1, 2, 3, 4, 5, 6$ as digits?

a) 5 b) 19 c) 7 d) None of these

10. 4 rectangles of same dimensions $x \times y$ are arranged in the following manner as shown in figure. Let A_1 be the area of the total square and A_2 be the area of the smaller square. Suppose $A_2 = \frac{1}{9}A_1$. Then $x : y$



a) 3:1 b) 2:1 c) 7:2 d) 5:2

11. Which of the following is true?

a) $2^{125} < 3^{75} < 5^{50}$, b) $3^{75} < 2^{125} < 5^{50}$, c) $5^{50} < 3^{75} < 2^{125}$, d) $2^{125} < 5^{50} < 3^{75}$.

12. Let x be a positive real number. Then

a) $x^2 + \pi^2 + x^{2\pi} > x\pi + (x+\pi)x^{\pi}$, b) $x^{\pi} + \pi^{x} > x^{2\pi} + \pi^{2x}$,
c) $x\pi + (x+\pi)x^{\pi} > x^2 + \pi^2 + x^{2\pi}$, d) None of these

13. Let $P(x) = (x-1)^{21} + (x-1)^{20}(1-x) + (x-1)^{19}(1-x)^2 + \cdots + (1-x)^{21}$. Then $P(2016)$ equals to

a) 0 b) 20^{2016}, c) 2016 d) 2015^{20}.

14. We define an operation $*$ as follows: $a * b = \frac{a-b}{1-ab}$. Then $1 * (2 * (3 * \cdots (2015 * 2016)) \cdots) =$

a) $2016 \times 2015 \times \cdots \times 1$, b) $\frac{1}{1-2016\times 2015}$, c) $\frac{2016}{2015}$, d) None of these

15. Let n be a two digit number such that

$$\text{sum of digits of } n + \text{ product of digits of } n = n$$

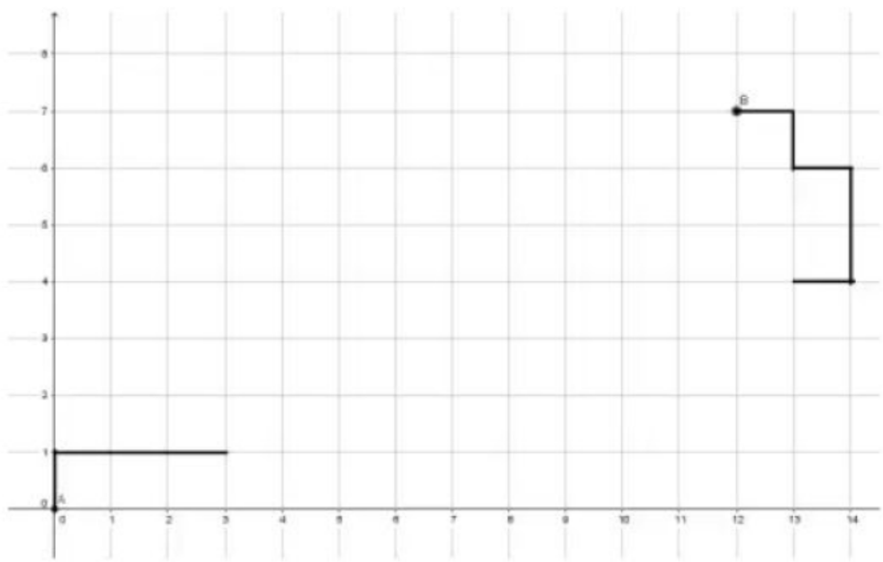
Then the unit digit of n is

a) 1 b) 9 c) 7 d) can't be determined

Short Answer Type Questions

[Each question carries a total of 15 marks. Credit will be given to partially correct answers]

1. Let ABC be a triangle with $AB = AC$. The bisector of $\angle ACB$ meet AB at M. Suppose $AM + MC = BC$. Show that $\angle BAC = 100°$
2. Two friends A and B are initially at points $(0, 0)$ and $(12, 7)$ respectively on the infinite grid plane (see figure). A takes steps of size 4 units and B takes steps of size 6 units along the grid lines. For example, a permissible step of A and B are shown in the figure [They are not necessarily the initial steps of A and B]. Show that it is not possible for them to meet at a point.



3. Mtrpia, a small country, has the following coins in circulation: 1 paisa, 2 paise, 5 paise, 10 paise, 20 paise, 50 paise, and 1 rupee. Suppose it is known that you can pay A paise with B coins. Prove that you can pay B rupees with A coins. [Assume that there are infinitely many coins of each type.]
4. Consider the following positive integers

$$a, a + d, a + 2d, a + 3d, \ldots$$

Suppose there is a perfect square in the above list of numbers. Then prove that there are infinitely many perfect squares in the above list.

5. 2016 coins are placed on a table with 50 heads up and remaining tails up. Suppose you are blindfolded and only thing you can do is flip the coins. Explain how you can separate the 2016 coins into two groups such that each group has equal number of heads.
6. Find all positive integers x and y such that $x, y, x + y$ and $x - y$ all are primes.

Use of calculators is not allowed. You may use a ruler and a compass for construction.

$\sim$ *Best of Luck* $\sim$

Mathematics Talent Reward Programme

Model Solutions for Junior Category

Multiple Choice Questions

[Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. (C) [volume varies with square of radius and varies with height]
2. (B) [Let centroid of ABC be G, then required area=$3 \times area(arc(ABG) - \Delta ABG)$]
3. (A) [Observe that the expression is square of area]
4. (B) [Observe that xy=$-\frac{1}{2}$, also see that $a_n = x^n + y^n = (x+y)a_{n-1} - xya_{n-2}$]
5. (A) [All the numbers end with 84 or 48, i.e. divisible by 4, aslo sum of digits divisible by 9]
6. (C) [Apply stars and bars theorem on which side is to be kept vertical]
7. (D) [Observe $x_{2i}x_{2i-1} = 2i$]
8. (A) [Take two type in a:b ratio and solve a/b]
9. (D) [Sum of digits of numbers formed by these digits =$21 \equiv 3 \pmod 9$, hence not perfect square]
10. (B) [$(x+y)^2 : (x+y)^2 - 4xy = 9 : 1$]
11. (C) [Take log]
12. (A) [$(x-\pi)^2 - (x - x^\pi)(x^\pi - \pi) = a^2 + b^2 + ab$, where $a = x - x^\pi, b = x^\pi - \pi$]
13. (A)[P(x)=0 $\forall x \in \mathbb{R}$]
14. (D) [Observe 1*(2*(... *(n))) ...)=1]
15. (B)[$10a + b = ab + a + b$, i.e. $b = 9$]

Short Answer Type Questions

[Each question carries a total of 15 marks. Credit will be given to partially correct answers]

1. Let D be a point on BC such that $CM = CD$. Then we have

$$AM + MC = BC = BD + CD = BD + CM \implies AM = BD$$

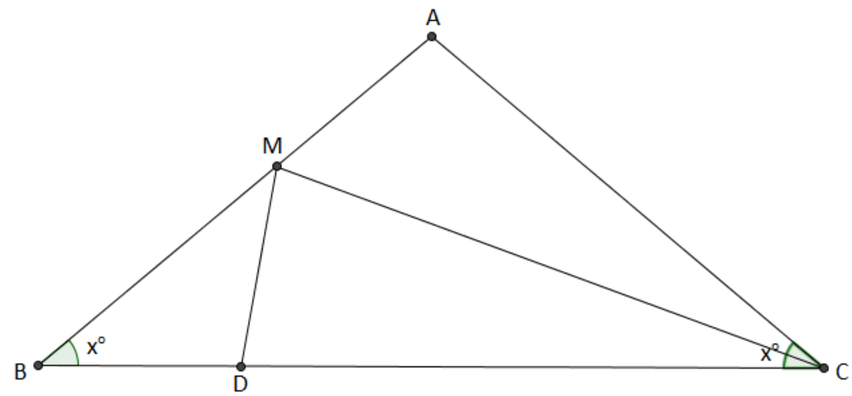
Now consider the triangles $\triangle BMD$ and $\triangle ABC$. We have $\angle MBD = \angle ACB$ and since CM is a bisector of $\angle ACB$, we have

$$\frac{AC}{BC} = \frac{AM}{BM} = \frac{BD}{BM}$$

Thus $\triangle BMD \sim \triangle ABC$. Let $\angle ABC = \angle ACB = x$ Then $\angle BMD = x$. Thus $\angle MDC = \angle BMD + \angle MBD = 2x$. Note that $CM = CD \implies \angle DMC = 2x$. Hence if we consider the angles of triangle $\triangle CMD$ we have

$$2x + 2x + \frac{x}{2} = 180^\circ \implies x = 40^\circ$$

This implies $\angle BAC = 180^\circ - 2 \times 40^\circ = 100^\circ$.



2. Consider the parity on the sum of the co-ordinates of postions of A and B separately and note that for each step (4 for A, 6 for B), the parity of the sum of the co-ordinates does not change. Hence A having sum of the co-ordinates 0 (even) initially and B having sum of the co-ordinates 19 (odd) initially can never meet.

3. Let x_i be the number of coins of i-th type used for paying A paise. Then we have

$$x_1 + x_2 + \cdots + x_7 = B, \quad x_1 + 2x_2 + 5x_3 + \cdots + 100x_7 = A$$

Observe that

$$\begin{aligned} 100B &= 100x_1 + 100x_2 + \cdots + 100x_7 \\ &= 100 \times x_1 + 50 \times 2x_2 + 20 \times 5x_3 + \cdots 1 \times 100x_7 \end{aligned}$$

Now if we define $y_1 = x_1, y_2 = 2x_2, y_3 = 5x_3, \ldots y_7 = 100x_7$ we have

$$y_1 + y_2 + \cdots + y_7 = A, \quad 100y_1 + 50y_2 + 20y_3 + \cdots + y_7 = 100B$$

Thus if we use $y_7 1$ paisa coins, $y_6 2$ paise coins, $y_5 5$ paise coins, $\ldots, y_1 1$ rupee coins, we can pay B rupees using A coins.

4. Suppose there is a square x^2 in that list. Observe that

$$(x + d)^2 = x^2 + 2xd + d^2 = x^2 + (2x + d)d$$

is of the form $x^2 + kd$ which must be in that list. Thus considering $x^2, (x + d)^2, (x + 2d)^2 \ldots$ we get a list of infinite perfect squares which is a sublist of the original list.

5. Take any 50 coins from 2016 coins to form heap A. The remaining 1966 coins form heap B say. Suppose there are x coins in heap A with heads facing up and hence there are $50 - x$ coins in heap A with tails facing up. If we flip all the coins of heap A, then we will get $50 - x$ coins of A with heads facing up. Note that there are $50 - x$ coins in heap B with heads facing up. This completes the proof.

6. Since $x - y$ is a prime, $x - y > 0 \Longrightarrow x > y$. Suppose both $x, y \geq 3$, then $x + y$ becomes even and hence not a prime. So one of them must be 2 . Hence $y = 2$ and $x \geq 3$. So we have $x - 2, x, x + 2$ as primes. Consider three cases:

Case 1: $x = 3k + 1$ where $k \geq 1$, then $x + 2 = 3k + 3 = 3(k + 1)$ which is certainly not a prime.
Case 2: $x = 3k + 2$ where $k \geq 1$, then $x - 2 = 3k$ which is prime only if $k = 1$. This forces $x = 5$. A simple checking shows that this is indeed a solution.

Case 3 : $x = 3k$ where $k \geq 1$, then $k = 1$, which forces $x - y = 1$, not a prime.

Thus $x = 5, y = 2$ is the only solution.

Mathematics Talent Reward Programme

Question Paper for Senior Category

17^{th} January,2016

Total Marks: 150 Allotted Time: 10:00 a.m. to 12:30 p.m.

Multiple Choice Questions

[You should answer these questions in the first page according to the order given in the question. Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. Sum of roots in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ of the equation $\sin x \tan x = x^2$ is

   a) $\frac{\pi}{2}$ b) 0 c) 1 d) None of these

2. Let f be a function satisfying $f(x+y+z) = f(x)+f(y)+f(z)$ for all integers x, y, z. Suppose $f(1) = 1$ and $f(2) = 2$. Then $\lim_{n\to\infty} \frac{1}{n^3} \sum_{r=1}^{n} 4rf(3r)$ equals

   a) 4 b) 6 c) 12 d) 24

3. z is a complex number and $|z| = 1$ and $z^2 \neq 1$. Then $\frac{z}{1-z^2}$ lies on

   a) a line not passing through origin b) $|z| = 2$
   c) x-axis d) y-axis

4. There are 168 primes below 1000 . Then sum of all primes below 1000 is

   a) 11555 b) 76127 c) 57298 d) 81722

5. ABCD is a quadrilateral on complex plane whose four vertices satisfy $z^4 + z^3 + z^2 + z + 1 = 0$. Then $ABCD$ is a

   a) Rectangle b) Rhombus c) Isosceles trapezium d) Square

6. Number of solutions of the equation $3^x + 4^x = 8^x$ in reals is

   a) 0 b) 1 c) 2 d) ∞

7. Let $\{x\}$ denote the fractional part of x. Then $\lim_{n\to\infty} \left\{(1+\sqrt{2})^{2n}\right\}$ equals

   a) 0 b) 0.5 c) 1 d) does not exist

8. Let p be a prime such that $16p + 1$ is a perfect cube. A possible choice for p is

   a) 283 b) 307 c) 593 d) 691

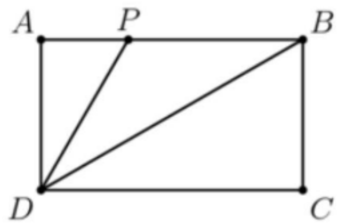
9. f be a function satisfying $2f(x) + 3f(-x) = x^2 + 5x$. Find $f(7)$.

   a) $-\frac{105}{4}$ b) $-\frac{126}{5}$, c) $-\frac{120}{7}$, d) $-\frac{132}{7}$.

10. Let $A = \{1, 2, \ldots, 100\}$. Let S be a subset of the power set of A such that any two elements of S has non zero intersection (Note that elements of S are actually some subsets of A). Then the maximum possible cardinality of S is

    a) 2^{99} b) $2^{99} + 1$ c) $2^{99} + 2^{98}$ d) None of these

11. In rectangle $ABCD, AD = 1, P$ is on $\overline{AB}$, and $\overline{DB}$ and $\overline{DP}$ trisect $\angle ADC$. What is the perimeter of



a) $3 + \frac{\sqrt{3}}{3}$, b) $2 + \frac{4\sqrt{3}}{3}$, c) $2 + 2\sqrt{2}$, d) $\frac{3+3\sqrt{5}}{2}$.

12. Let $f(x) = (x-1)(x-2)(x-3)$. Consider $g(x) = \min\{f(x), f'(x)\}$. Then the number of points of discontinuity are

a) 0 b) 1 c) 2 d) more than 2

13. Let $P(x) = x^2 + bx + c$. Suppose $P(P(1)) = P(P(-2)) = 0$ and $P(1) \neq P(-2)$. Then $P(0) =$

a) $-\frac{5}{2}$ b) $-\frac{3}{2}$ c) $-\frac{7}{4}$ d) $\frac{6}{7}$

14. Let $[x]$ denotes the greatest integer less than or equal to x. Find x such that $x[x[x[x]]] = 88$

a) π b) 3.14 c) $\frac{22}{7}$ d) All of these

15. Suppose $50x$ is divisible by 100 and kx is not divisible by 100 for all $k = 1, 2, \ldots, 49$. Find number of solutions for x when x takes values $1, 2, \ldots, 100$.

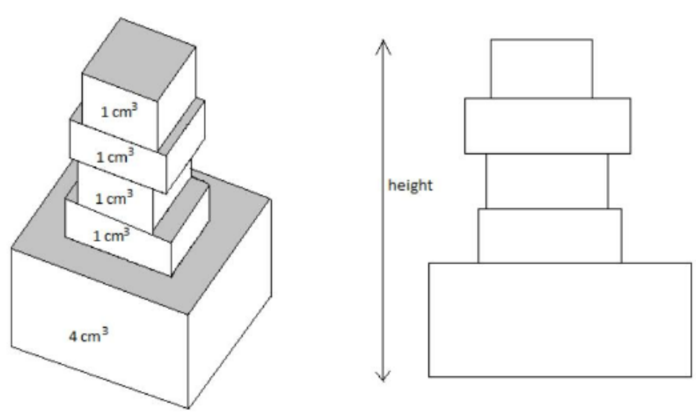
a) 20 b) 25 c) 15 d) 50

Short Answer Type Questions

[Each question carries a total of 15 marks. Credits will be given to partially correct answers]

1. Show that there exist a polynomial $P(x)$ whose one coefficient is $\frac{1}{2016}$ and remaining coefficients are rational numbers, such that $P(x)$ is an integer for any integer x.

2. 5 blocks of volume $1\text{ cm}^3, 1\text{ cm}^3, 1\text{ cm}^3, 1\text{ cm}^3$, and 4 cm^3 are placed one above another to form the structure as shown in the figure. Suppose the sum of surface areas of upper face of each block is 48 cm^2. Determine the minimum possible height of the whole structure.



Structure

Front view of the structure

3. Prove that for any positive integer n there are n consecutive composite numbers all less than 4^{n+2}. [You may use the fact that product of all primes, which are less than k, is less than 4^k and this holds for all positive integers k.]

4. For any given k points in a plane, we define the diameter of the points as the maximum distance between any two points among the given points. Suppose n point are there in a plane with diameter d. Show that we can always find a circle with radius $\frac{\sqrt{3}}{2}d$ such that all the points lie inside the circle.

5. Let $\mathbb{N}$ be the set of all positive integers. $f, g : \mathbb{N} \to \mathbb{N}$ be functions such that f is onto, g is one-one and $f(n) \geq g(n)$ for all positive integers n. Prove that $f = g$.

6. Consider the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. A partition Π of A is a collection of disjoint sets whose union is A. For example, $\Pi_1 = \{\{1, 2\}, \{3, 4, 5\}, \{6, 7, 8, 9\}\}$ and $\Pi_2 = \{\{1\}, \{2, 5\}, \{3, 7\}, \{4, 6, 8, 9\}\}$ can be considered as partitions of A. For each Π partition, we consider the function π defined on the elements of A. $\pi(x)$ denotes the cardinality of the subset in Π which contains x. For example, in case $\Pi_1, \pi_1(1) = \pi_1(2) = 2, \pi_1(3) = \pi_1(4) = \pi_1(5) = 3$, and $\pi_1(6) = \pi_1(7) = \pi_1(8) = \pi_1(9) = 4$. For Π_2 we have, $\pi_2(1) = 1$, $\pi_2(2) = \pi_2(5) = 2, \pi_2(3) = \pi_2(7) = 2$, and $\pi_2(4) = \pi_2(6) = \pi_2(8) = \pi_2(9) = 4$. Given any two partitions Π and Π', show that there are two numbers x and y in A, such that $\pi(x) = \pi'(x)$ and $\pi(y) = \pi'(y)$. [Hint: Consider the case where there is a block of size greater than or equal to 4 in a partition and the alternative case.]

Use of calculators is not allowed. You may use a ruler and a compass for construction.

~ Best of Luck ~

Model Solutions for Senior Category

Multiple Choice Questions

[Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. (B) [Observe that if a is a solution, then $-a$ is a solution]
2. (A) [$f(n+2) = f(n) + 2$, by induction $f(n) = n$]
3. (D) [$\frac{z(1-\bar(z))}{|1-z^2|^2}$]
4. (B) [There are 167 odd primes, hence sum is odd, moreover $p_n > n$,i.e. sum¿$\frac{167.168}{2}$]
5. (C)[Any 4 vertices of a regular pentagon makes an isosceles trapezium]
6. (B) [Take $f(x) = (\frac{3}{8})^x + (\frac{4}{8})^x$, hence $f(0) > 1, f(1) < 1$, and f is monotonically decreasing.]
7. (C) [observe $a_n = (\sqrt{2}-1)^n+(\sqrt{2}+1)^n$ is an integer]// 8. (B) [let $16p+1 = n^3$, i.e. $16p = (n-1)(n^2+n+1)$. Now $n^2 + n + 1$ is odd, hence equals 1 or p. Also $n - 1 < n^2 + n + 1$,hence n-1=16]
9. (B) [replace x with -x and solve for f(x)]
10. (A) [No. of subsets consisting 1 is 2^{99}, hence $|S| \geq 2^{99}$. Now consider the set of subsets which contain 1, and set of there compliments. These are disjoint and has cardinality $2^{99} + 1$. Now, if a set of $2^{99} + 1$ subset had that property, then by PHP, it would consisted (X,X^c).]
11. (B)
12. (D) [sketch the graphs]
13. (A) [P(1) and P(-2) are two roots of the quadratic, now observe the sum of roots]
14. (C) [Observe that x is rational]
15. (B) [x is even but not divisible by 5]

Short Answer Type Questions

[Each question carries a total of 15 marks. Credit will be given to partially correct answers]

1. Consider $P(x) = \frac{1}{2016}(x-1)(x-2)\cdots(x-2016)$. Clearly all coefficients of $P(x)$ are rationals. Observe that the leading coefficient of $P(x)$ is $\frac{1}{2016}$. Note that product of 2016 consecutive integers is always divisible by 2016 . Hence this $P(x)$ is our required polynomial.

2. <u>First Solution</u>: Let $a_1, a_2, \ldots, a_5$ be the surface areas of upper face of the blocks. We have $a_1 + a_2 + \cdots + a_5 = 48$. Note that the heights of the blocks are given by

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}, \text{ and } \frac{4}{a_5}$$

Note that by Cauchy-Schwarz inequality

$$\begin{aligned}
&48\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{4}{a_5}\right)\\
&=\left(\frac{1}{(\sqrt{a_1})^2}+\frac{1}{(\sqrt{a_2})^2}+\frac{1}{(\sqrt{a_3})^2}+\frac{1}{(\sqrt{a_4})^2}+\left(\frac{2}{\sqrt{a_5}}\right)^2\right)\left((\sqrt{a_1})^2+\cdots+(\sqrt{a_5})^2\right)\\
&\geq (1+1+1+1+2)^2 = 36
\end{aligned}$$

Hence

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{4}{a_5} \geq \frac{36}{48} = \frac{3}{4}$$

Equality holds when $a_5 = 2a_1 = 2a_2 = 2a_3 = 2a_4 = 16$. Thus minimum possible height is certainly $\frac{3}{4}$.

<u>Second solution:</u> One can arrive at the same conclusion using AM-HM inequality. Observe that

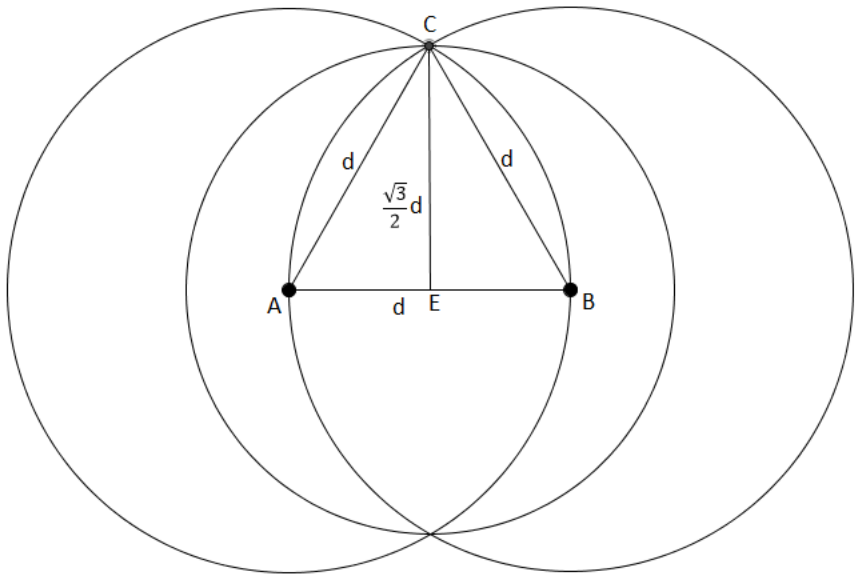
$$
\begin{aligned}
&\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{4}{a_5}\\
&=\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{1}{a_5/2}+\frac{1}{a_5/2}\\
&\geq\frac{36}{a_1+a_2+a_3+a_4+a_5/2+a_5/2}=\frac{36}{48}=\frac{3}{4}
\end{aligned}
$$

3. Let P be the product of all primes which are less than or equal to $n+1$. Consider the following n consecutive numbers

$$P+2, P+3, \ldots, P+(n+1)$$

We will now show that $P+k$ is composite for any $k \in \{2, 3, \ldots n+1\}$. Consider any prime divisor of k say u. Clearly u divides P. Thus u divides $P+k$ implying $P+k$ is composite. This completes the proof.

4. Consider the two points say A and B whose distance is d. Now all points must lie within the circle centre at A as well as the circle centre at B. So they must lie within the intersection of the two circles. Consider the circle drawn with centre as the mid point of A and B and with radius the altitude of the shown equilateral triangle ABC. Note that the radius equals $\frac{\sqrt{3}}{2}d$. This new circle covers the intersection region of the two circles and hence all the points within it.



5. We shall prove the statement $P(k)$: For every $k \in \mathbb{N}$ there exists a unique $x_k \in \mathbb{N}$ such that $f(x_k) = g(x_k) = k$ by induction on k. Since f is onto, there exists $x_1 \in \mathbb{N}$ such that $f(x_1) = 1$. But $g \leq f$ so $g(x_1) = 1$. Since g is one-one this x_1 is unique. Thus we have proved $P(1)$. Now let $P(k)$ be true. We shall prove that $P(k+1)$ is true. As f is onto, there exists $x_{k+1} \in \mathbb{N}$ such that $f(x_{k+1}) = k+1$. But $g \leq f$ and by induction hypothesis g already takes all values less that $k+1$. So $g(x_{k+1}) = k+1$. Since g is one-one this x_{k+1} is unique. Thus by the principle of mathematical induction, the statement $P(k)$ holds for all natural numbers k. Observe that $P(k)$ implies $f = g$. This completes the proof.

6. Part 1: Let us assume for the sake of contradiction there do not exist two distinct elements satisfying the property.

Part 2: If there were 4 distinct values of π or π' in decreasing, the first value is at least 4 , the second value is at least 3 , the third value is at least 2 , and the fourth value is at least 1 , so we need at least 4 elements in the first partition, 3 in the second partition, 2 in the third partition, and 1 in the fourth partition, which implies we need at least 10 elements. But $4+3+2+1>9$. Hence there can only be at most 3 distinct values for $\pi(x)$ and $\pi'(x)$.

Part 3: In addition, only at most three elements can share a value for $\pi(x)$ or $\pi'(x)$ If there were 4 elements which had the same value for $\pi(x)$, WLOG $\pi(1)=\pi(2)=\pi(3)=\pi(4)$, then by Part 1 we know $\pi'(1), \pi'(2), \pi'(3), \pi'(4)$ must all be different, but this contradicts Part 2 . So for each value of $\pi(x)$, only at most three elements can share that value.

So we know from Part 3 that $\pi(x)$ cannot equal 4 . It follows that the only possible way is for 3 elements to satisfy $\pi(x)=3,3$ other elements to satisfy $\pi(x)=2$, and 3 other elements to satisfy $\pi(x)=1$. But it's impossible for exactly three elements to satisfy $\pi(x)=2$, because the number of elements satisfying $\pi(x)=2$ must be even. So we have a contradiction.

Mathematics Talent Reward Programme

Question Paper for Junior Category

15^{th} January,2017

Total Marks: 102 Allotted Time: 2:00 p.m. to 4:30 p.m.

Multiple Choice Questions

[You should answer these questions in the first page according to the order given in the question. Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. The number of ordered pairs (a, b) of natural numbers such that $a^b + b^a = 100$ is

   a) 1 b) 2 c) 3 d) 4

2. $ABCD$ be a rectangle. E and F are the midpoints of BC and CD respectively. The area of $\triangle AEF$ is 3 sq units. The area of rectangle $ABCD$ is

   a) 4 b) 6 c) 8 d) 16

3. Suppose a, b, c are three distinct integers from 2 to 10 (both inclusive). Exactly one of ab, bc and ca is odd and abc is a multiple of 4 . The arithmetic mean of a and b is an integer and so is the arithmetic mean of a, b and c. How many such (unordered) triplets are possible?

   a) 4 b) 5 c) 6 d) 7

4. $PQRS$ is a rectangle in which $PQ = 2016PS$. T and U are the midpoints of PS and PQ respectively. QT and US intersect at V. Suppose

$$R = \frac{\text{Area of triangle PQT}}{\text{Area of quadrilateral QRSV}}$$

   $R =$

   a) $\frac{5}{12}$ b) $\frac{2016}{2017}$ c) $\frac{2}{7}$ d) $\frac{3}{8}$

5. For any three real numbers a, b, and c, with $b \neq c$, the operation $\otimes$ is defined by:

$$\otimes(a, b, c) = \frac{a}{b - c}$$

   What is $\otimes(\otimes(1, 2, 3), \otimes(2, 3, 1), \otimes(3, 1, 2))$?

   a) $-\frac{1}{2}$ b) $-\frac{1}{4}$ c) $\frac{1}{2}$ d) $\frac{1}{4}$

6. A company sells peanut butter in cylindrical jars. Marketing research suggests that using wider jars will increase sales. If the diameter of the jars is increased by 25% without altering the volume, by what percent must the height be decreased?

   a) 10%, b) 25% c) 36% d) 64%

7. Let

$$V_1 = \frac{7^2 + 8^2 + 15^2 + 23^2}{4} - \left(\frac{7 + 8 + 15 + 23}{4}\right)^2$$
$$V_2 = \frac{6^2 + 8^2 + 15^2 + 24^2}{4} - \left(\frac{6 + 8 + 15 + 24}{4}\right)^2$$
$$V_3 = \frac{5^2 + 8^2 + 15^2 + 25^2}{4} - \left(\frac{5 + 8 + 15 + 25}{4}\right)^2$$

   Then

a) $V_3 < V_2 < V_1$, b) $V_3 < V_1 < V_2$, c) $V_1 < V_2 < V_3$ d) $V_2 < V_3 < V_1$.

8. How many natural numbers, less than 2017 , are divisible by 3 but not by 5 ?

   a) 548 b) 538 c) 528 d) None of these

9. Consider 3 numbers, 4, 6 and 10. In 1st step we choose any a, b from the 3 numbers and replace them with $\frac{3a-4b}{5}$ and $\frac{4a+3b}{5}$ to get a new triplet of numbers and again perform the operation on new triplet and so on. How many distinct ways are there to obtain 4,6 and 12 as a triplet for the first time?

   a) 3 b) 5 c) 7 d) None of these

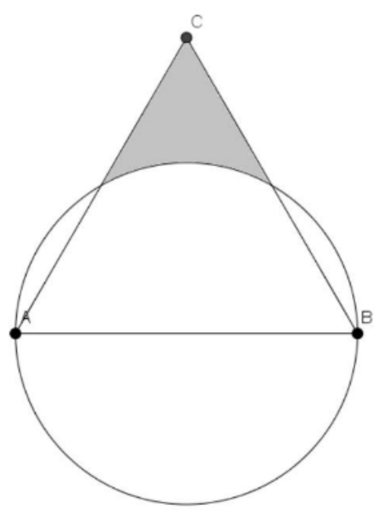
10. Let a and b be relatively prime integers with $a > b > 0$ and $\frac{a^3-b^3}{(a-b)^3} = \frac{73}{3}$. What is $a - b$?

    a) 1 b) 3 c) 9 d) 27

Short Answer Type Questions

[Each question carries a total of 12 marks. Credit will be given to partially correct answers]

1. Let ABC be an equilateral triangle constructed on the diameter AB of circle of radius 1 as a side. Find the area of the shaded portion with justification.

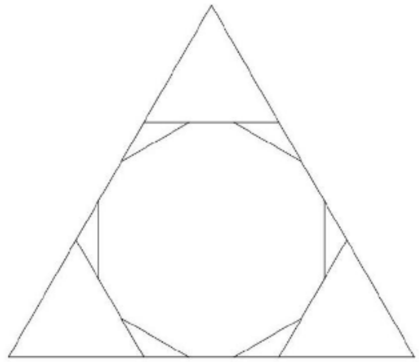


2. There are 30 balls in a box. You have to write one number in each ball. However the only numbers you are allowed to write are 0,1 or 4 . Let X be the number obtained by adding all the numbers on the balls. Find all possible values of X with justification.

3. Find all primes p and q such that $p + q = (p - q)^3$. Justify your answer.

4. The natural number y is obtained from the number x by rearranging its digits. Suppose $x + y = 10^{200}$. Prove that x is divisible by 10 .

5. Consider an equilateral triangle of area 1 . We call the triangle P_0. We find the trisecting points of each side of P_0 and cutoff the corners to form a new polygon (in fact a hexagon) say P_1 as shown in figure. We again trisect each side of the hexagon and cutoff the corners to form polygon P_2, with 12 sides, as shown in the figure. Find the area of P_2.



6. The numbers $1, 3, 5, 7, 2, 4, 6, 8$ are written in a row on a blackboard (in the given order). Two players A and B play the following game by making moves. In each move, a player picks two consecutive numbers written in the board, say a and b, and replace it by $a + b$ or $a - b$ or $a \times b$. Note that after each move there is one less number on the blackboard. Suppose player A makes the first move. The first player wins if the final result after 7 moves is odd, and loses otherwise. Show that no matter what player 1 does, player 2 can always win i.e., player 2 has a winning strategy.

Mathematics Talent Reward Programme

Model Solutions for Junior Category

Multiple Choice Questions

[Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. (D) [Assume $b \geq a \implies 2a^a \leq 50$, only possible values for a is $1,2$]
2. (C)[Let a, b be the sides of rectangle. Now find the area of three right angled triangles in terms of a, b]
3. (A)[Notice that exactly 2 of a, b, c are odd and $4|c \implies c = 4, 8.$]
4. (D)[Consider P as $(0,0)$ and PQ, PS as X, Y axis respectively. Find the coordinate of V]
5. (B)
6. (C)[Compute volume in both condition and then equate]
7. (C)[Note in each case only 2 no's are changed and the value of the later part which is subtracted is same for V_1, V_2, V_3]
8. (B)[No. of natural numbers less than 2017 that are divisible by $k = \lfloor \frac{2017}{k} \rfloor$]
9. (D)[Note that $a^2 + b^2 + c^2$ is invariant i.e. remains same in all step.]
10. (B)[Note $3 \cdot 73ab = 70(a^2 + b^2 + ab)$ and a, b are relatively prime $\implies ab\ |70$ and $70\ |ab$]

Short Answer Type Questions

[Each question carries a total of 15 marks. Credit will be given to partially correct answers]

1. let us draw the center O and join OD and OE as shown in figure. Area $ABC = \sqrt{3}$ Note that $OD = OA$ as they are radii of the same cirle and $\angle DAO = 60°$ Hence OAD is equilateral triangle with side length 1 . Hence Area $DAO = \frac{\sqrt{3}}{4}$. Similarly Area $EBO = \frac{\sqrt{3}}{4}$. Note $\angle DOE = 60°$. Hence the area of sector DOE is $\frac{\pi}{6}$. So the area of the shaded region is $\sqrt{3} - \frac{\sqrt{3}}{2} - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$.

2. We denote the triplet (x, y, z) to denote that there are x number of four-balls y number of oneballs and z number of zero-balls. We first show that X can take any integer value from 0 to 111 . Take any number from 0 to 111 say t. We write it as $t = 4q + r$ where $r = 0, 1, 2, 3$. Note that if $(q, r, 30 - q - r)$ is observed, X can indeed achieve t. We just have to ensure $q, r, 30 - q - r$ are all non-negative numbers. Note that $r \leq 3$ and since $t \leq 111$, it implies $q \leq 27$. Thus $30 - q - r \geq 0$. Hence X can take any integer value from 0 to 111 . Note that any multiple of 4 say $4q$, less than or equal to 120 , is possible if $(q, 0, 30 - q)$ observed. We note that $X = 113$ corresponds to $(28, 1, 1), X = 114$ to $(28, 2, 0)$ and $X = 117$ to $(29, 1, 0)$.

   We will now show that X cannot take values $115, 118, 119$. If $X = 115$ corresponds to (x, y, z), then $x + y + z = 30$ and $115 = 4x + y \leq 4x + y + z = 3x + 30$ which implies $3x \geq 85 \Longrightarrow x \geq 29$ as x is an integer, but then $4x + y \geq 4x \geq 116$, so $X = 115$ is not possible. Similarly if $X \geq 118$,

   $$118 \leq X = 4x + y \leq 3x + 30 \Longrightarrow 3x \geq 88 \Longrightarrow x \geq 30$$

   Hence it forces $X = 120$. Hence X can take any integer values between 0 to 120 except 115, 118 and 119 .

3. Suppose p, q leaves same remainder when divided by 3 . Then $p - q$ is divisible by 3 . But $p + q$ is not divisible by 3 unless both p, q are divisible by 3 which forces $p = q = 3$ (as they are primes). This clearly does not gives us a solution. Thus p, q leaves different remainders when divisible by 3. If both are not 3 , then one of them leaves remainder 1 and the other leaves 2 when divided by 3 . Then $p + q$ is divisible by 3 but $p - q$ does not. Hence no solution is possible. This forces that one of them must be 3 . Clearly $p = 3$ implies $q < 3$ as $(3 - q)^3 = 3 + q > 0$. But this forces $q = 2$ which does not satisfies the equation. Hence $q = 3$. Thus the equation becomes

   $$p + 3 = (p - 3)^3 = p^3 - 9p^2 + 27p - 27 \Longrightarrow p\left(p^2 - 9p + 26\right) = 30$$

   Hence p must divides 30 . The only possibility is $p = 5$. On checking we see that $p = 5, q = 3$ indeed satisfies the equation.

4. Note 10^{200} has 201 digits. Then x must have atmost 200 digits. If x has 199 digits then y have atmost 199 digits. Hence their sum cannot be a 201 digit number. Thus x has exactly 200 digits. Let $x_1x_2\cdots x_{200}$ be the decimal representation of x and $y_1y_2\cdots y_{200}$ be the decimal representation of y. Suppose x is not divisible by 10 . Then $x_{200}\neq 0$. Thus $y_{200}=10-x_{200}$. Thus on adding the unit digits we carry 1 to the ten's digit. Thus $x_{199}+y_{199}$ must be 9. Again we carry 1 to the next digit and so on. Thus we arrive that $x_i+y_i=9$ for all $i<200$ and $x_{200}+y_{200}=10$. Hence

$$x_1+x_2+\cdots+x_{200}+y_1+y_2+\cdots+y_{200}=9\times 199+10=\text{ odd}$$

But y is just a rearrangement of x. Hence sum of digits of x plus sum of digits of y must be even. This gives us a contradiction. Hence x is divisible by 10 .

5. We label some of the points of figure as shown below. We join the diagonal of FE of the hexagon P_1. Observe that $AE : AC = 1:3$ and $AD : AB = 1:3$ and hence $DE\|BC$. Thus $\triangle ADE$ and $\triangle ABC$ are similar. Hence by properties of similar triangles Area ADE : Area $ABC = 1:9$. Hence Area $ADE=\frac{1}{9}$. Similarly area of other similar 'corners' are $\frac{1}{9}$. Hence area of hexagon is $1-\frac{3}{9}=\frac{2}{3}$. We now focus on the corners inside the hexagon. Note that $DG : DF = 1:3$ and $DH : DE = 1:3$ and hence $GH\|FE$. Thus $\triangle DGH$ and $\triangle DFE$ are similar. Hence by properties of similar triangles Area DGH : Area $DFE = 1:9$. Again DE is the median of triangle AEF. Hence Area DFE = Area $ADE=\frac{1}{9}$. Thus Area $DGH=\frac{1}{81}$. Similarly area of other 5 small corners are $\frac{1}{81}$. Hence Area of P_2 equals $\frac{2}{3}-\frac{6}{81}=\frac{2}{3}-\frac{2}{27}=\frac{16}{27}$

6. Only the fact that whether the numbers are odd (O) or even (E) is important for the problem. Note that at any stage if all the numbers are even, player 1 can never win. Player 2 target would be to leave two even number after penultimate move. Suppose we start with $OOEE$ sequence on board. If player 1 has any hope to win, he must not convert the odds into even. Hence the possible sequence after player 1 move is OOE or OEE, then on adding first two terms or multiplying first two terms player 2 wins.

   For the general $OOOOEEEE$ problem, we will try to reach $OOEE$ or $EEEE$. For this 8string, we either (i) try to maintain the initial symmetry of odds and evens or (ii) reduce the number of odds but keeping then separated from even numbers. If in any step player 1 chooses 2 evens to modify, no of odds remain same, even numbers decreases by 1 . Then player 2 can take 2 odd numbers and multiply them and reach case (i). Similar steps can be taken if player 1 takes 2 odds and multiplies them. If player 1 takes 2 odd numbers and get an even then two case can occur:

   (a) There is an even number adjacent to the one player 1 got in the step; take two evens and do any operation

   (b) There is an odd number adjacent to it, then the string has EO in it. Transform it to O It's easy to see this converts the previous string to strings mentioned in (i) and (ii),

   Finally in the case player 1 chooses OE : (a) $OE\to E$: We take two even and transform it to even (b) $OE\to O$: We take two odd and transform them to odd.

Mathematics Talent Reward Programme

Question Paper for Senior Category

15^{th} January,2017

Total Marks: 102 | Allotted Time: 10:00 a.m. to 12:30 p.m.

Multiple Choice Questions

[You should answer these questions in the first page according to the order given in the question. Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. The number of real solutions of the equation $(9/10)^x = -3 + x - x^2$ is

   a) 2 b) 0 c) 1 d) None of these

2. $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} =$

   a) $\sqrt{e}$, b) ∞ c) Does not exist d) None of these

3. Let $p(x) = x^4 - 4x^3 + 2x^2 + ax + b$. Suppose that for every root λ of $p, 1/\lambda$ is also a root of p. Then $a + b =$

   a) -3 b) -6 c) -4 d) -8

4. Let $F_1 = F_2 = 1$. We define inductively $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 2$. Then the sum

$$F_1 + F_2 + F_3 + \cdots + F_{2017}$$

   is

   a) even but not divisible by 3 b) odd but divisible by 3
   c) odd and leaves remainder 1 when divided by 3 d) NOne of these

5. Compute the number of ordered quadruples of positive integers (a, b, c, d) such that

$$a! \cdot b! \cdot c! \cdot d! = 24!$$

   a) 4 b) 6 c) 4^4 d) None of these

6. Let $p(x)$ is a polynomial of degree 4 with leading coefficients 1 . Suppose $p(1) = 1, p(2) = 2, p(3) = 3$ and $p(4) = 4$. Then $p(5) =$

   a) 5 b) $\frac{25}{6}$ c) 29 d) 35

7. Let $ABCD$ be a quadrilateral with sides $AB = 2, BC = CD = 4$ and $DA = 5$. The opposite angles A and C are equal. The length of diagonal BD equals

   a) (A) $2\sqrt{6}$, b) (B) $3\sqrt{3}$, c) (C) $3\sqrt{6}$, d) (D) $2\sqrt{3}$.

8. How many finite sequences $x_1, x_2, \ldots, x_m$ are there such that $x_i = 1$ or 2 and $\sum_{i=1}^{m} x_i = 10$?

   a) 89 b) 73 c) 107 d) 119

9. From a point P outside of a circle with centre O, tangent segments PA and PB are drawn. If $\frac{1}{OA^2} + \frac{1}{PA^2} = \frac{1}{16}$. Then $AB =$

a) 4 b) 6 c) 8 d) 10

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $\lim_{x\to\infty} f'(x) = 1$, then

a) f is increasing, b) f is unbounded, c) f' is bounded, d) All of these

Short Answer Type Questions

[Each question carries a total of 12 marks. Credits will be given to partially correct answers]

1. A monic polynomial is a polynomial whose highest degree coefficient is 1 . Let $P(x)$ and $Q(x)$ be monic polynomials with real coefficients, and $\deg P(x) = \deg Q(x) = 10$. Prove that if the equation $P(x) = Q(x)$ has no real solutions, then $P(x+1) = Q(x-1)$ has a real solution.

2. Let a, b, c be positive reals such that $a + b + c = 3$. Show that
$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \leq \frac{6}{\sqrt{(a+b)(b+c)(c+a)}}$$

3. Let $f : [0,1] \to [0,1]$ be a continuous function. We say $f \equiv 0$ if $f(x) = 0$ for all $x \in [0,1]$ and similarly $f \not\equiv 0$ if there exist at least one $x \in [0,1]$ such that $f(x) \neq 0$. Suppose $f \not\equiv 0, f \circ f \not\equiv 0$, but $f \circ f \circ f \equiv 0$. Do there exist such an f ? If yes construct such an function, if no prove it. [Note that $f \circ f(x) = f(f(x))$ and $f \circ f \circ f(x) = f(f(f(x)))$.]

4. 4. An irreducible polynomial is a non-constant polynomial that cannot be factored into the product of two non-constant polynomials. Consider the following statements:

   Statement 1: $p(x)$ be any monic irreducible polynomial with integer coefficients and degree ≥ 4. Then $p(n)$ is prime for at least one natural number n.

   Statement 2: $n^2 + 1$ is prime for infinitely many values of natural number n.

   Show that if Statement 1 is true then Statement 2 is also true.

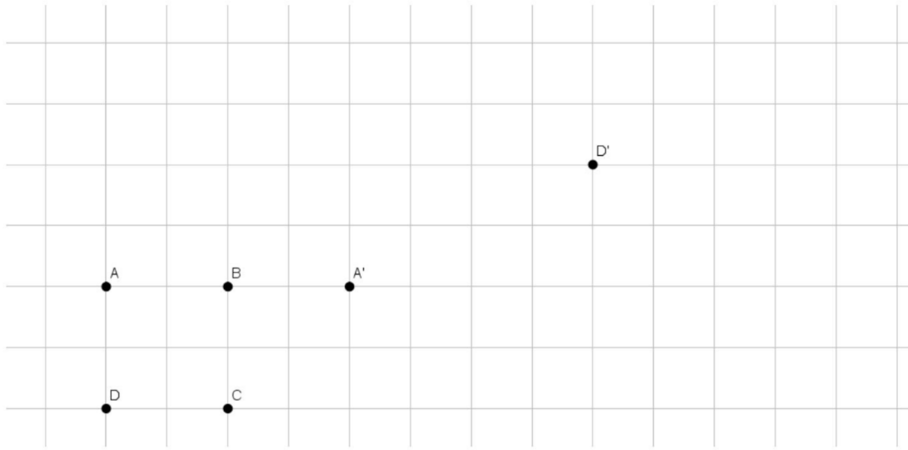
5. Let $\mathbb{N}$ be the set of all natural numbers. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijective function. Show that there exist three numbers a, b, c in arithmetic progression such that $f(a) < f(b) < f(c)$.

6. Let us consider an infinite grid plane as shown below. We start with 4 points A, B, C, D, that form a square, as shown below.

   We perform the following operation: We pick two points say X and Y from the current points. X is reflected about Y to get X'. We remove X and add X' to get a new set of 4 points and treat it as our current points.

   For example in the figure suppose we choose A and B (we can choose any other pair too). Then reflect A about B to get A'. We remove A and add A' Thus A', B, C, D is our new 4 points. We may again choose D and A' from the current points. Reflect D about A to obtain D' and hence A', B, C, D' are now new set of points. Then similar operation is performed on this new 4 points and so on.

   Starting with A, B, C, D, can you get a bigger square by some sequence of such operations?



Use of calculators is not allowed. You may use a ruler and a compass for construction.

~ Best of Luck ~

Mathematics Talent Reward Programme

Model Solutions for Senior Category

Multiple Choice Questions

[Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. (B)[Find the signs of RHS and LHS]
2. (D) [Note the limit is of the form 1^∞, $\lim_{x\to 0}(\frac{sinx}{x})^{\frac{1}{x^2}} = e^{\lim_{x\to 0}\frac{1}{x^2}\cdot(\frac{sinx}{x}-1)}$]
3. (A)[Note in this case sum of the roots = sum of the roots taken 3 at a time]
4. (C)[Fibonacci numbers are periodic modulo 2 with period $1,1,0$ and they are also periodic modulo 3 with period $1,1,2,0,2,2,1,0$]
5. (D)[Note 23 is a prime] 6. (C)[Consider the roots of $h(x) = p(x) - x$]
7. (A)[Recall the cosine rule]
8. (A)[Let there are a terms with $x_i = 1$ and b terms with $x_i = 2$]
9. (C)[Use Pythagoras theorem and equate area of triangle OAP]
10. (B)

Short Answer Type Questions

[Each question carries a total of 12 marks. Credit will be given to partially correct answers]

1. Let $P(x) = x^{10} + a_9x^9 + a_8x^8 + \cdots + a_0$ and $Q(x) = x^{10} + b_9x^9 + b_8x^8 + \cdots + b_0$. Let $R(x) = P(x) - Q(x)$ Note that the equation $R(x) = (a_9 - b_9)\,x^9 + (a_8 - b_8)\,x^8 + \cdots + (a_0 - b_0)$ If $a_9 \neq b_9$, then $R(x)$ is of degree 9 , then the polynomial $R(x)$ must have a real root which contradicts the assumption that $R(x) = P(x) - Q(x) = 0$ has no real solutions. Thus $a_9 = b_9$.

   Let $S(x) = P(x+1) - Q(x-1)$. Then

   $$S(x) = (x+1)^{10} - (x-1)^{10} + a_9(x+1)^9 - a_9(x-1)^9 + T(x)$$

   where $T(x)$ is polynomial of degree at most 8 . Clearly $a_9\left[(x+1)^9 - (x-1)^9\right]$ is of degree atmost 8 since on expansion x^9 coefficient cancels out, whereas

   $$\begin{aligned}(x+1)^{10} - (x-1)^{10} &= \left[x^{10} + 10x^9 + A(x)\right] - \left[x^{10} - 10x^9 + B(x)\right]\\ &= 20x^9 + A(x) - B(x)\end{aligned}$$

   where $A(x)$ and $B(x)$ are polynomials of degree at most 8 . Hence $S(x)$ is of degree exactly equal to 9 hence it must have a real root. Thus $P(x+1) - Q(x-1)$ has real solution.

2. Note that the given inequality can be written as

   $$\sqrt{a(a+b)(a+c)} + \sqrt{b(b+c)(b+a)} + \sqrt{c(c+a)(c+b)} \leq 6$$

   Note that

   $$\begin{aligned}(a+b+c)^2 - 3(ab+bc+ca) &= a^2 + b^2 + c^2 - ab - bc - ca\\ &= \frac{1}{2}(a-b)^2 + \frac{1}{2}(b-c)^2 + \frac{1}{2}(c-a)^2 \geq 0\end{aligned}$$

   Hence $(a+b+c)^2 \geq 3(ab+bc+ca)$. Since $a+b+c = 3$, we have that $ab+bc+ca \leq 3$.

   Solution 1:

By Cauchy Schwarz inequality we have that

$$(a+b+c)((3a+bc)+(3b+ca)+(3c+ab)) \geq (\sqrt{a(3a+bc)}+\sqrt{b(3b+ca)}+\sqrt{c(3c+ab)})^2$$

Note that $a+b+c=3$ and hence $3a+bc=(a+b+c)a+bc=(a+b)(a+c), 3b+ca=(b+c)(b+a)$, and $3c+ab=(c+a)(c+b)$. Thus taking square roots in the above inequality we have

$$\sqrt{3(3a+3b+3c+ab+bc+ca)} \geq \sqrt{a(a+b)(a+c)}+\sqrt{b(b+c)(b+a)}+\sqrt{c(c+a)(c+b)}$$

Note that $ab+bc+ca \leq 3$, hence

$$\sqrt{a(a+b)(a+c)}+\sqrt{b(b+c)(b+a)}+\sqrt{c(c+a)(c+b)} \leq \sqrt{3(3\times 3+3)}=\sqrt{36}=6$$

Solution 2: Observe that by AM-GM inequality we have

$$\frac{7a+bc}{2}=\frac{4a+(3a+bc)}{2} \geq \sqrt{4a(3a+bc)}$$

Since $a+b+c=3, 3a+bc=(a+b+c)a+bc=(a+b)(a+c)$. Hence $\sqrt{4a(a+b)(a+c)} \leq \frac{1}{2}(7a+bc)$. We divide both sides by 2 to obtain

$$\sqrt{a(a+b)(a+c)} \leq \frac{1}{4}(7a+bc)$$

Analogously we obtain

$$\sqrt{b(b+c)(b+a)} \leq \frac{1}{4}(7b+ca)$$
$$\sqrt{c(c+a)(c+b)} \leq \frac{1}{4}(7c+ab)$$

Adding all three inequalities we have

$$\begin{aligned}
&\sqrt{a(a+b)(a+c)}+\sqrt{b(b+c)(b+a)}+\sqrt{c(c+a)(c+b)}\\
&\leq \frac{1}{4}(7a+bc)+\frac{1}{4}(7b+ca)+\frac{1}{4}(7c+ab)\\
&= \frac{1}{4}(7(a+b+c)+(ab+bc+ca))\\
&\leq \frac{1}{4}(7\times 3+3)=6
\end{aligned}$$

The last inequality is due to the fact that $a+b+c=3$ and $ab+bc+ca \leq 3$.

3. There exists such a function satisfying all conditions. We construct one such function.

We first show how to get an f such that $f \not\equiv 0$ but $f \circ f \equiv 0$ Let $A=\{x \in [0,1] \mid f(x)=0\}$ and $B=\{x \in [0,1] \mid f(x) \neq 0\}$ be the set where f takes value non zero. Since $f(f(x))$ is zero for all $x, f(x)$ must take values in A. If we take $A=[0,1/2]$, we have to ensure that f is continuous, $f(x)>0$ for all $x>1/2$ and $f(x) \leq \frac{1}{2}$ for all x. To do this we may take f as a part of a line whose slope is sufficiently small so that $f(x) \leq \frac{1}{2}$ for all x. For example we may take $f(x)=\left(x-\frac{1}{2}\right)$ for $x \geq \frac{1}{2}$. Note that continuity is maintained and $f \not\equiv 0$ and $f(x) \leq \frac{1}{2}$ for all x. Hence $f \circ f \equiv 0$.

To do the part where $f \not\equiv 0, f \circ f \not\equiv 0$ but $f \circ f \circ f \equiv 0$, we apply the same idea, however the time we increase the slope so that $f \circ f \not\equiv 0$. Suppose $f(x)=0$ for $x \leq 1/2$ and $f(x)=k\left(x-\frac{1}{2}\right)$

for $x \geq 1/2$ where $1 < k < 2$. Then $f(f(1)) = f(k/2) \neq 0$ as $\frac{k}{2} \geq \frac{1}{2}$. So $f \circ f \not\equiv 0$. To ensure $f \circ f \circ f \equiv 0$, we note that f is a non decreasing function hence it attains maximum at $x = 1$. But $f(f(1)) = f(k/2) = \frac{1}{2}k(k-1)$ which is less than $\frac{1}{2}$ as long as we choose sufficiently close to 1. We may choose $k = \frac{3}{2}$ for example for this purpose. Then $f(f(1)) = \frac{3}{8} < \frac{1}{2}$. Then $f \circ f$ is always less than $\frac{1}{2}$ which forces $f \circ f \circ f \equiv 0$. So the following functions works:

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1/2] \\ \frac{3}{2}\left(x - \frac{1}{2}\right) & \text{if } x \in (1/2, 1] \end{cases}$$

4. Suppose statement 1 is true but statement 2 is false. Hence $n^2 + 1$ is prime for finitely many values of n. Hence there exist $N > 0$ such that $n^2 + 1$ is composite for all $n > N$. Let $p(x) = x^4 + 1$. It is easy to check that $p(x)$ is irreducible. Note that this implies $p(x + k)$ is irreducible for any k. Finally we consider $p(x + N) = (x + N)^4 + 1$. Note that statement 2 implies $p(n + N)$ is prime at least for one natural number n. Hence suppose $p(n_0 + N)$ is prime for some natural number n_0, But $p(n_0 + N) = \left((n_0 + N)^2\right)^2 + 1$ and $(n_0 + N)^2 > N^2 \geq N$ which is a contradiction to the fact that $n^2 + 1$ is always composite after $n > N$.

5. Since f is bijection let us choose a such that $f(a) = 1$. If $f(a) < f(a+1) < f(a+2)$ we are done. If not the only other possibility is $f(a) < f(a+2) < f(a+1)$. This implies $f(a+2)$ lies between $f(a+1)$ and $f(a)$. We then consider $a, a+2, a+4$. If $f(a) < f(a+2) < f(a+4)$ we are done. If not the only other possibility is $f(a) < f(a+4) < f(a+2)$. Since $f(a+2) < f(a+1), f(a+4)$ lies between $f(a+1)$ and $f(a)$. We then consider $a, a+4, a+8$. If $f(a) < f(a+4) < f(a+8)$ does not hold, we can again conclude that $f(a+8)$ lies between $f(a+1)$ and $f(a)$ and so on.

   Note that there are finitely many natural numbers between $f(a+1)$ and 1 . Since f is a bijection, only for finitely many values of $n, f(n)$ lies between $f(a+1)$ and 1 . So if we continue in the above fashion we must get a n_0 such that $f(a) < f\left(a + 2^{n_0}\right) < f\left(a + 2^{n_0+1}\right)$.

6. Let $ABCD$ be the initial square. Suppose it is possible to reach a bigger square say $EFGH$. Note that it is not necessary that sides of $EFGH$ is parallel to the grid lines.

   We claim that the operations are reversible i.e., if starting from P, Q, R, S you reach P', Q', R', S' then you can come back to P, Q, R, S by some sequence of operations. To prove this, consider two points X and Y. Suppose we reflect X about Y to get X'. Then according to the rule we remove X and add X'. We can now reflect X' about Y to get back X. We may remove X' and add X to get back the two points. Thus such an operation is reversible. Clearly for a sequence of operations, we may apply the reversibility of each operations to get the reversed sequence of operations. This proves our claim.

   So there is a sequence of operations by which starting from $EFGH$ one can reach smaller square $ABCD$. Now without loss of generality assume $E = (0,0), F = (0,1), G = (1,1)$ and $H = (1,0)$.

   We now claim that every point that can arise by this operation has integer co-ordinates. This is true because if $X = (x_1, x_2)$ and $Y = (y_1, y_2)$, then $X' = (2y_1 - x_1, 2y_2 - x_2)$. So inductively every such point must have integer co-ordinates.

   $ABCD$ is smaller square then $EFGH$. According to the co-ordinate system that we impose, $EFGH$ has side length 1. So, $ABCD$ has side length less than 1. But A, B, C, D must have all integer co-ordinates. Since any two distinct points having integer co-ordinates is at least 1 distance apart, this gives us a contradiction.

Mathematics Talent Reward Programme

Question Paper for Junior Category

14^{th} January, 2018

Total Marks: 100 Allotted Time: 2:00 p.m. to 4:30 p.m.

Multiple Choice Questions

[You should answer these questions in the first page according to the order given in the question. Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. The number of solutions of the equation $\frac{5}{2-x}+\frac{x-5}{x+2}+\frac{3x+8}{x^2-4}=0$ is

   a) 2 b) 1 c) 0 d) None of these

2. How many triangles ABC are there such that $\angle A = 60°, AB = 3$ cm and $BC = 2$ cm ?

   a) 1 b) 2 c) 6 d) None of these

3. Let $3 \mid a^2+a+1$, where a is an integer. Which of the following is false ?

   a) $3 \mid a^2-1$, b) 3 does not divide $a+1$,
   c) a^3+2 is not divisible by 3 , d) None of these

4. Which of the following is true?

   a) Last digit of 18^{19} is prime, b) Last digit of 17^7 is 7
   c) Last digit of 17^9 is not 7 d) None of these

5. How many ordered pairs (a, b) of rational numbers are there such that $\frac{a}{b}+\frac{b}{a}=3$?

   a) 1 b) 2 c) Infinitely many d) None of these

6. Let ABC be an isosceles triangle with $AB = BC$ and $\angle B = 20°$. Then which of the following is true ?

   a) $AB > 3AC$, b) $2AC < AB < 3AC$, c) $AC < AB < 2AC$, d) NOne of these

7. What is the product of the real roots of the equation $x^2+18x+30 = 2\sqrt{x^2+18x+45}$?

   a) 40 b) 20 c) $\sqrt{30}$ d) None of these

Short Answer Type Questions

[Each question carries a total of 12 marks. Credits will be given to partially correct answers]

1. Let $ABCD$ be a parallelogram. Let E be the midpoint of CD. Let O be the intersecting point of the lines AE and BD. Find $AO : OE$.

2. How many ordered triplets of integers (a, b, c) are there such that $a^3bc = 24$?

3. There are 5 pairs of balls which are kept in 5 different boxes (i.e. each box has 2 identical balls). In one of the boxes both the balls weigh $9g$ each. In the remaining 4 boxes all balls weigh $8g$ each. You have a weighing machine with 2 pans. If you put some balls on the left pan and some on the right, its reading will show you the (value of the weight on the left pan - value of weight on the right pan). Find the minimum number of times the weighing machine needs to be used in order to identify the balls which are heavier.

4. How many natural numbers are there such that the sum of the digits of the number equals the product of the digits of the number in its decimal representation? (Note that the sum and product of a single digit number is the number itself.)

5. Prove that there exist infinitely many perfect squares starting with the digits " 2018 ".

6. Prove that there do not exist 2 natural numbers n_1, n_2 such that 9^{n_1} and 9^{n_2} are palindrome numbers and the difference in the number of digits of 9^{n_1} and 9^{n_2} is 2017. (Note: A palindromic number is a number that remains the same when its digits are reversed, for example $313, 121, 2332$ are palindrome numbers.)

Use of calculators is not allowed. You may use a ruler and a compass for construction.

$\sim$ *Best of Luck* $\sim$

Mathematics Talent Reward Programme

Model Solutions for Junior Category

Multiple Choice Questions

[Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. (A)
2. (D)[Recall the cosine rule]
3. (C)[Note that $3 \nmid a$]
4. (C)[Last digit of 17^{19} = Last digit of 7^{19} and so on]
5. (D)[Take $x = \frac{a}{b}$, then x is irrational]
6. (B)[Recall sin rule]
7. (B)[Take $a = x^2 + 18x$]

Short Answer Type Questions

[Each question carries a total of 12 marks. Credit will be given to partially correct answers]

1. Consider the triangles ΔDOE and ΔBOA. We have that $\angle EDO = \angle OBA$, $\angle DEO = \angle OAB$ and $\angle DOE = \angle AOB$. Hence, we can say that $\Delta DOE \sim \Delta BOA$ (AAA criterion of similarity). Hence we have
$$\frac{EO}{AO} = \frac{DE}{AB} = \frac{1}{2}$$

2. 24 may be factored as,
$$24 = 2^3 \cdot 3$$
Possible choices of a are $1, 2$.

   For $a = 2$, $bc = 3$. Since 3 isn't a perfect square, number of (b, c) tuples is the number of divisors of 3, that is, $(1+1) = 2$(Obtained from the prime factorisation of $3 = 3^1$)

   For $a = 1$, $bc = 24$. Number of tuples=$(3+1)\cdot(1+1) = 8$ $(24 = 2^3 \cdot 3)$

   Number of possible ordered triplets of positive integers= $2 + 8 = 10$

   Now, changing the sign of any two keeps the product same. We have $\binom{3}{2} = 3$ ways of changing the sign of two positive integers in an ordered triplet.

   So, number of ordered possible triplets of integers= $10 \cdot (3+1) = 10 \cdot 4 = \boxed{40}$

3. The minimum number of times the weighing machine needs to be used is $\boxed{2}$.

   At first, we pick a ball from each of the boxes.

   In the first step, we weigh any two of these balls. If they weigh the same, we omit these and proceed to the second step. Else, the heavier one weighs 9g.

   For the second step, we weigh any two of the remaining three weighs 9g. Else, the heavier one between the two weighs 9g.

4. There are infinitely many such integers. The main fact that we use is if there are 1's in the decimal representation of a number, then they do not alter the product of the digits but increase the sum of the digits. We give an explicit construction of an infinite set of numbers which have the product of digits equal to the sum of digits. Note that the numbers stated below aren't all such numbers with the aforesaid property.

   Define
$$I_k = \underbrace{222\cdots2}_{k \text{ many 2's}}\ \underbrace{111\cdots1}_{2^k-2 \text{ many 1's}}$$
   Check that all I_k for $k \geq 3$ satisfies the required property.

5. For a perfect square to start with 2018, it has to satisfy the following :

$$2018 \cdot 10^k \leq n^2 \leq 2019 \cdot 10^k$$

for some $k \in \mathbb{N}$.

We may restrict k to be even for convenience, and obtain:

$$2018 \cdot 10^{2k} \leq n^2 \leq 2019 \cdot 10^{2k}, k \in \mathbb{N}$$

$$\implies \sqrt{2018} \cdot 10^k \leq n \leq \sqrt{2019} \cdot 10^k$$

$$\sqrt{2019} - \sqrt{2018} = \frac{1}{\sqrt{2019} + \sqrt{2018}} \geq \frac{1}{45 + 45} = \frac{1}{90}$$

So, for $k \geq 2$, $\left(\sqrt{2019} - \sqrt{2018}\right) 10^k > 1$ and thus, there exists a perfect square between $2018 \cdot 10^{2k}$ and $2019 \cdot 10^{2k}$ for all $k \geq 2$, implying that there are infinitely many such perfect squares.

6. Since the difference between the number of digits is 2017, one of them has to have an even number of digits.

Now, for a palindrome having $2k$ number of digits,

$$N = \sum_{i=0}^{2k-1} 10^i a_i = \sum_{i=0}^{k-1} \left(10^i + 10^{2k-1-i}\right) a_i$$

Since $2k - 1$ is odd, for any $i, 0 \leq i \leq k - 1$, $i \neq 2k - 1 - i$ and so, the term with parentheses is a multiple of 11, impossible for a power of 9. Hence, we conclude that there are no two such natural numbers.

Mathematics Talent Reward Programme

Question Paper for Senior Category

14^{th} January,2018

Total Marks: 100 Allotted Time: 10:00 a.m. to 12:30 p.m..

Multiple Choice Questions

[You should answer these questions in the first page according to the order given in the question. Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. A coin is tossed 9 times. There are 2^9 possible outcomes. In how many of these outcomes does no two successive heads occur?

   a) 55 b) 34 c) 89 d) None of these

2. $\lim\limits_{x\to 0^+} \dfrac{[x]}{\tan(x)} =$

   a) -1 b) 1 c) 0 d) does not exist

3. Let F_n denote the Fibonacci sequence such that $F_1 = 0, F_2 = 1, F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 3$. Then $\sum\limits_{n=3}^{\infty} \dfrac{18 + 999F_n}{F_{n-1}F_{n+1}} =$

   a) 2016 b) 2017 c) 2018 d) None of these

4. In $\triangle ABC, O$ is an interior point such that $\angle BOC = 90^\circ, \angle CAO = \angle ABO, \angle BAO = \angle BCO$. Then $\frac{AC}{OC} =$

   a) $\sqrt{2}$ b) 2 c) $\sqrt{\frac{3}{2}}$ d) None of these

5. Let M and m denote the maximum value and the minimum value of the function $f(x) = \cos\left(x^{2018}\right)\sin(x)$ in the interval $[-2\pi, 2\pi]$ respectively, then $m + M =$

   a) $\frac{1}{2}$ b) $-\frac{1}{\sqrt{3}}$ c) $\frac{1}{2018}$ d) None of these

6. In a class of 80 students, 40 are male and 40 are female. Also, exactly 50 students wear glasses. Then which of the following is true?

   a) Exactly 10 boys wear glasses b) At least 20 girls wear glasses
   c) At most 25 boys do not wear glasses d) At most 30 girls do not wear glasses

7. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. How many functions $f : A \to A$ can be defined such that $f(1) < f(2) < f(3)$?

   a) $\begin{pmatrix} 8 \\ 3 \end{pmatrix}$, b) $\begin{pmatrix} 8 \\ 3 \end{pmatrix} 5^8$ c) $\begin{pmatrix} 8 \\ 3 \end{pmatrix} 8^5$ d) $\frac{8!}{3!}$.

Short Answer Type Questions

[Each question carries a total of 12 marks. Credits will be given to partially correct answers]

1. If x, y, z are real numbers such that $x < y < z$, prove that

$$(x-y)^3 + (y-z)^3 + (z-x)^3 > 0$$

2. Let $P(x)$ be a polynomial with real coefficients such that $P(n)$ is an integer for any integer n. Prove that the coefficients of $P(x)$ must be rational.

3. Does there exist a continuous function f, such that $f(f(x)) = -x^{2019} \quad \forall x \in \mathbb{R}$?

4. Let S be a finite subset of $\mathbb{R}$. Let f be a function from S to S such that $|f(x_1) - f(x_2)| \leq \frac{1}{2}|x_1 - x_2|$ $\forall x_1, x_2 \in S$. Prove that $f(x) = x$ for some $x \in S$.

5. (a) Prove that the sequence of remainders obtained when the Fibonacci numbers are divided by n is periodic, where $n \in \mathbb{N}$.
(b) Prove that there does not exist a non-constant polynomial $P(x)$ with integer coefficients such that $P(F_n)$ is prime for all $n \in \mathbb{N}$, where F_n denotes the nth term of the Fibonacci sequence.

6. Let $d(n)$ be the number of divisors of n. Prove that we can colour the natural numbers using 2 colours such that if for an infinite increasing sequence $\{a_1, a_2, a_3, \cdots\}$, the sequence $\{d(a_1), d(a_2), \cdots\}$ is a non-constant geometric progression, then all the terms $\{a_1, a_2, a_3 \cdots\}$ cannot have the same colour. (You may use the fact that we can colour the natural numbers using 2 colours such that all the terms of any infinite increasing A.P cannot have the same colour.)

Use of calculators is not allowed. You may use a ruler and a compass for construction.
~ Best of Luck ~

Mathematics Talent Reward Programme

Model Solutions for Senior Category

Multiple Choice Questions

[Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. (C)[Find the no of ways to choose k non-consecutive places out of 9 places and put H in those selected places]
2. (C) $\left[\dfrac{\lfloor x\rfloor}{\tan x} = 0 \forall x \in (0,1)\right]$
3. (A)$\left[\frac{1}{F_{n-1}F_{n+1}} = \frac{1}{F_{n-1}F_n} - \frac{1}{F_nF_{n+1}}\right]$ and $\left[\frac{F_n}{F_{n-1}F_{n+1}} = \frac{1}{F_{n-1}} - \frac{1}{F_{n+1}}\right]$
4. (A)[Extend OC to C' such that O is midpoint of CC'.Conclude that $AOBC'$ is cyclic. Now, $\triangle C'AC \sim \triangle AOC$]
5. (D)[$f(x)$ is an odd function]
6. (D)
7. (B)[First choose 3 no's out of 8 for $f(1), f(2), f(3)$]

Short Answer Type Questions

[Each question carries a total of 12 marks. Credit will be given to partially correct answers]

1. Call $y - x = a > 0$, $z - y = b > 0$. Then $z - x = a + b$. We have to show that

$$-a^3 - b^3 + (a+b)^3 = 3ab(a+b) > 0$$

which is trivially true since both a and b are strictly positive.

2. We have $P(i) = a_i$ for $i = 1, 2, \dots d+1$, where d is the degree of the polynomial, and $a_1, a_2, \dots a_{d+1}$ are integers. Hence, by Lagrange's interpolation, we may obtain a d degree polynomial Q with rational coefficients s.t. $Q(i) = a_i$ for $i = 1, 2, \dots d+1$. As P and Q are both d degree polynomials, $P - Q$ is at most a d degree polynomial and has roots $1, 2, \dots d+1$, i.e. d+1 roots, and so is identically 0. Hence $P \equiv Q$. Since Q has rational coefficients, we conclude that P has rational coefficients as well.

3. Let, if possible, there exists a continuos function f such that

$$f(f(x)) = -x^{2019} \qquad \forall x \in \mathbb{R}$$

Clearly f is bijective (and hence one-one) and continuos which implies that f is monotone.

Case 1: Assume that f is monotonically increasing. Therefore, we have that

$$x > y \implies f(x) > f(y) \implies f(f(x)) > f(f(y)) \implies -x^{2019} > -y^{2019}$$

which is absurd.

Case 2: Assume that f is monotonically decreasing. Therefore, we have that

$$x > y \implies f(y) > f(x) \implies f(f(x)) > f(f(y)) \implies -x^{2019} > -y^{2019}$$

which is again absurd.

Hence, we conclude that no such f can exist.

4. Let $\{s_i : 1 \leq i \leq n\}$ be an enumeration of S. The set S has n elements. Let,

$$d = \min\{|s_i - s_j| : 1 \leq i < j \leq n\}$$

.

Now, for distinct $x, y \in S$,

$$|f^n(x) - f^n(y)| \leq \frac{1}{2}|f^{n-1}(x) - f^{n-1}(y)| \leq \dots \frac{1}{2^n}|x - y|$$

So, for large enough n,

$$\frac{1}{2^n}|x - y| < d$$

$\forall\ x, y \in S$.

Then, for this n, $\forall x, y \in S$,

$$f^n(x) = f^n(y) = k$$

.

We show that k is a fixed point.

$$k = f^n(f(x)) = f(f^n(x)) = f(k)$$

So, k is a fixed point.

5. **(a)** For two consecutive remainders in the sequence, we have n^2 choices(Set of remainders is $\{0, 1, 2, \dots, n-1\}$ upon division by n).

   Since the Fibonacci sequence has infinitely many terms , we shall have some pair of consecutive remainders repeat. Let $F_{k_1} \equiv F_{k_2} \pmod n$ and $F_{k_1+1} \equiv F_{k_2+1} \pmod n$ for some $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$.

   Verify that $F_n \equiv F_{n+(k_2-k_1)}\ \forall n \in \mathbb{N}$ using the property $F_{n+2} = F_{n+1} + F_n$.

   (b) Let $F_n = p$ for some $n \in \mathbb{N}$. From (a), it may be concluded that, $\exists k \in \mathbb{N}$ such that $\forall m \in \mathbb{N}$,

$$F_{m+k} \equiv F_m \pmod p$$

   .

   Now, for $a \in \mathbb{N}$, $F_{an} \equiv F_n \pmod p$ and so,

$$P(F_{an}) \equiv P(F_n) \pmod p$$

   Since $P(F_{an})$ is a prime, $P(F_{an}) = p\ \forall a \in \mathbb{N}$. Implying that the polynomial given by, $Q(x) = P(x) - p$ has infinitely many distinct roots and hence, is identically zero. Therefore, P is a constant polynomial, a contradiction.

6. We construct one such colouring.

   Each natural number has at least two divisors, that is, 1 and the number itself. So, we colour the numbers green for which $2^{(2k-1)^2} < d(n) \leq 2^{(2k)^2}$ and color the rest red.

   Let r be the common ratio of the geometric progression. Let $n \in \mathbb{N}$ be such that $2^n > r$.

   Consider the largest index i such that $d(a_i) \leq 2^n$.

   Now, note that, $d(a_i) > 2^{(n-1)^2}$ since otherwise we shall have,

$$d(a_i + 1) = rd(a_i) < 2^n \cdot 2^{(n-1)^2} = 2^{n^2-n+1} < 2^{n^2}$$

   , contradicting the maximality of i.

   A latter inequality is strict owing to $n > 1$ ($2^n > r > 1$).

   Observe that,

$$2^{n^2} < d(a_{i+1}) = rd(a_i) < 2^n \cdot 2^{n^2} < 2^{(n+1)^2}$$

   Now, since $2^{(n-1)^2} < d(a_i) \leq 2^{n^2}$ and $2^{n^2} < d(a_{i+1}) < 2^{(n+1)^2}$, a_i and a_{i+1} are of different colours.