Mathematics Talent Reward Programme

Question Paper for Senior Category

18^{th} January,2015

Total Marks: 150 Allotted Time: 10:00 a.m. to 12:30 p.m.

Multiple Choice Questions

[You should answer these questions in the first page according to the order given in the question. Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. How many distinct arrangements are possible for wearing five different rings in the five fingers of the right hand? (We can wear multiple rings in one finger)

   a) $\frac{10!}{5!}$, b) 5^5, c) $\frac{9!}{4!}$, d) None of these.

2. Let $f_n(x) = \underbrace{xx\cdots x}_{n \text{ times}}$ that is, $f_n(x)$ is an n digit number with all digits x, where $x \in \{1, 2, \ldots, 9\}$. Then which of the following is $(f_n(3))^2 + f_n(2)$?

   a) $f_n(5)$ b) $f_{2n}(9)$, c) $f_{2n}(1)$, d) None of these.

3. If $A_i = \frac{x-a_i}{|x-a_i|}, i = 1, 2, \ldots n$ for n numbers $a_1 < a_2 < \cdots < a_m < \cdots < a_n$, then $\lim_{x\to a_m} (A_1A_2\ldots A_n) =$?

   a) $(-1)^{m-1}$, b) $(-1)^m$, c) 1 , d) Does not exist.

4. Let n be an odd integer. Placing no more than one **X** in each cell of a $n \times n$ grid, what is the greatest number of **X** 's that can be put on the grid without getting n**X** 's together vertically, horizontally or diagonally?

   a) $2\binom{n}{2}$, b) $\binom{n}{2}$, c) $2n$, d) $2\binom{n}{2} - 1$.

5. How many integral solutions are there for the equation $x^5 - 31x + 2015 = 0$?

   a) 2 b) 4 c) 1 d) None of these.

6. Let AC and CE be perpendicular line segments, each of length 18 . Suppose B and D are the midpoints of AC and CE respectively. If F be the point of intersection of EB and AD, then the area of $\triangle BDF$ is?

   a) $27\sqrt{2}$, b) $18\sqrt{2}$, c) 13.5 , d) 18

7. How many x are there such that $x, [x], \{x\}$ are in harmonic progression (i.e, the reciprocals are in arithmetic progression)? (Here $[x]$ is the largest integer less than equal to x and $\{x\} = x - [x]$)

   a) 0 b) 1 c) 2 d) 3

8. In $\triangle ABC, AB = AC$ and D is foot of the perpendicular from C to AB and E the foot of the perpendicular from B to AC, then

   a) $BC^3 > BD^3 + BE^3$, b) $BC^3 < BD^3 + BE^3$,

   c) $BC^3 = BD^3 + BE^3$, d) None of these.

9. How many 5×5 grids are possible such that each element is either 1 or 0 and each row sum and column sum is 4 ?

   a) 64 b) 32 c) 120 d) 96

10. If $\sum_{i=1}^{n} \cos^{-1}(\alpha_i) = 0$, then find $\sum_{i=1}^{n} \alpha_i$.

   a) $\frac{n}{2}$, b) n c) nπ d) $\frac{n\pi}{2}$.

11. $S = \{1, 2, \ldots, 6\}$. Then find out the number of unordered pairs of (A, B) such that $A, B \subseteq S$ and $A \cap B = \emptyset$.

   a) 360 b) 364 c) 365 d) 366

12. The maximum value of $\sin^4\theta + \cos^6\theta$ will be?

   a) $\frac{1}{2\sqrt{2}}$, b) $\frac{1}{2}$, c) $\frac{1}{\sqrt{2}}$, d) 1.

13. Define $f(x) = \max\{\sin x, \cos x\}$. Find at how many points in $(-2\pi, 2\pi)$, $f(x)$ is not differentiable?

   a) 0 b) 2 c) 4 d) ∞

14. $z = x + iy$ where x and y are two real numbers. Find the locus of the point (x, y) in the plane, for which $\frac{z+i}{z-i}$ is purely imaginary (that is, it is of the form ib where b is a real number). [Here, $i = \sqrt{-1}$]

   a) A straight line b) A circle c) A parabola d) None of these

15. Find out the number of real solutions of $x^2 e^{\sin x} = 1$

   a) 0 b) 1 c) 2 d) 3

Short Answer Type Questions

[Each question carries a total of 15 marks. Credits will be given to partially correct answers]

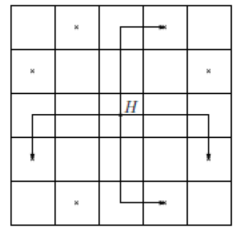
1. In a room there is a series of bulbs on a wall and corresponding switches on the opposite wall. If you put on the n-th switch the n-th bulb will light up. There is a group of men who are operating the switches according to the following rule: they go in one by one and starts flipping the switches starting from the first switch until he has to turn on a bulb; as soon as he turns a bulb on, he leaves the room. For example the first person goes in, turns the first switch on and leaves. Then the second man goes in, seeing that the first switch on turns it off and then lights the second bulb. Then the third person goes in, finds the first switch off and turns it on and leaves the room. Then the fourth person enters and switches off the first and second bulbs and switches on the third. The process continues in this way. Finally we find out that first 10 bulbs are off and the 11-th bulb is on. Then how many people were involved in the entire process?

2. Let x, y be numbers in the interval $(0, 1)$ such that for some $a > 0, a \neq 1$

$$\log_x a + \log_y a = 4 \log_{xy} a.$$

   Prove that $x = y$.

3. Let $n > 3$ be an integer. Show that, in an $n \times n$ chessboard, it is possible to traverse to any given square from another given square using a knight. (A knight can move in a chessboard by going two steps in one direction and one step in a perpendicular direction as shown in the given figure.)

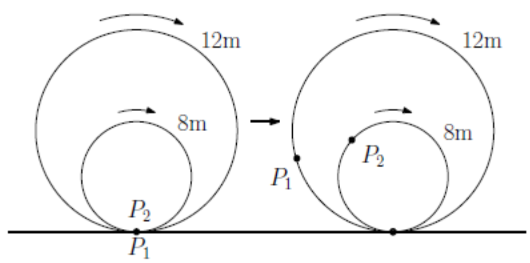


4. Find all real numbers $x_1, x_2, \ldots, x_n$ satisfying,

$$\sqrt{x_1 - 1^2} + 2\sqrt{x_2 - 2^2} + \cdots + n\sqrt{x_n - n^2} = \frac{1}{2}(x_1 + x_2 + \cdots + x_n)$$

5. Let a be the smallest and A the largest of n distinct positive integers. Prove that the least common multiple of these numbers is greater than or equal to na and that the greatest common divisor is less than or equal to $\frac{A}{n}$.

6. In the following figure, the bigger wheel has circumference 12 m and the inscribed wheel has circumference 8 m. P_1 denotes a point on the bigger wheel and P_2 denotes a point on the smaller wheel. Initially P_1 and P_2 coincide as in the figure. Now we roll the wheels on a smooth surface and the smaller wheel also rolls in the bigger wheel smoothly. What distance does the bigger wheel have to roll so that the points will be together again?



Use of calculators is not allowed. You may use a ruler and a compass for construction.

~ Best of Luck ~

Mathematics Talent Reward Programme

Model Solutions for Senior Category

Multiple Choice Questions

[Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. (C) [Arrange rings in 10! ways, then find the number of nonnegative integer solutions of $a+b+c+d+e=10$]
2. (C) [Observe $f_n(k)=kf_n(1)$, hence $(f_n(3))^2+f_n(2)=(9f_n(1)+2)f_n(1)=f_{2n}(1)$]
3. (D)[right hand limit and left hand limit have opposite signs]
4. (A)[Placing X everywhere except a diagonal is a valid construction, i.e. n(n-1) X's. Also, putting more than n(n-1) X's makes atleast one row full by PHP]
5. (D) [0 solutions]
6. (C) [triangle formed by centroid and two vertices has $\frac{1}{3}$rd area of actual triangle, and median divides area by half]
7. (B) [$\frac{1}{x}+\frac{1}{\{x\}}=\frac{2}{[x]}$, then put $\{x\}=x-[x]$, get $[x]=\sqrt{2}(x-[x])$]
8. (A) [Observe $BC^2=BD^2+BE^2$,$BC>BD, BC>BE$,hence $BC^3=(BD^2+BE^2)(BC)>BD^3+BE^3$]
9. (C) [We have 6-i choices of placing the 0 in each row]
10. (B) [$cos^{-1}(x)\geq 0$, hence $a_i=\frac{\pi}{2}\forall i$]
11. (C) [Chose k elements for A, chose B in 2^{6-k} ways, sum and divide by 2. Then add the case (ϕ,ϕ)]
12. (D) [Check the maxima]
13. (C) [sinx and cosx graph intersect each other at 4 points]
14. (B) [$\bar{a}=-a$ when a is imaginary]
15. (C) [$f(x)=x^2e^{sinx}$, then f(0)=0, f($\frac{5\pi}{2}$)=f($\frac{-5\pi}{2}$)=$\frac{25\pi^2}{4}>1$, hence atleast two solutions, and observe there are exactly two solutions]

Short Answer Type Questions

[Each question carries a total of 15 marks. Credit will be given to partially correct answers]

1. When the n-th person leaves the room, the bulbs can be represented as a binary number, say A_n whose k-th digit from right is 1 if k-th bulb is ON and 0 if k-th bulb is OFF. Denote by $(B)_{10}$ to be the decimal representation of a binary number B. Then we will show that $(A_n)_{10}=n$.

   Clearly when the 1 st person leaves, only the first switch is turned on. Therefore $A_1=1\Longrightarrow(A_1)_{10}=1$. Now we use induction. Suppose it $(A_n)_{10}=n$ holds for all $n\leq m$. We will show $(A_{m+1})_{10}=m+1$. If the 1st switch is turned off when the $(m+1)$-th person enters, then $(A_{m+1})_{10}-(A_m)_{10}=1\Longrightarrow(A_{m+1})_{10}=m+1$. If the first r switches are turned on and $(r+1)$-th switch is turned off when the $(m+1)$-th person enters, then

   $$(A_{m+1})_{10}-(A_m)_{10}=(\dots 1\underbrace{00\dots 0}_{r \text{ times}})_{10}-(\dots 0\underbrace{11\dots 1}_{r\text{ times}})_{10}=2^r-\sum_{k=0}^{r-1}2^k=1$$

   Hence by induction the claim is true. We use this method to compute the number of involved people. Now note that our configuration gives A_n to be a binary number with 1 followed by 10 0 's. So, n, the number of persons is

   $$(\underbrace{100\dots 0}_{10\text{ times}})_{10}=2^{10}=1024$$

2. From the given equation we have

$$\begin{aligned}
&\log_x a + \log_y a = 4\log_{xy} a \\
\implies &\frac{\log a}{\log x} + \frac{\log a}{\log y} = 4\frac{\log a}{\log xy} \\
\implies &\log a\left(\frac{1}{\log x} + \frac{1}{\log y}\right) = \frac{4\log a}{\log x + \log y}
\end{aligned}$$

Since $a > 0$ and $a \neq 1$, this ensures $\log a \neq 0$. Hence we can cancel $\log a$ both sides to get

$$\begin{aligned}
&\frac{1}{\log x} + \frac{1}{\log y} = \frac{4}{\log x + \log y} \\
\implies &2 + \frac{\log y}{\log x} + \frac{\log x}{\log y} = 4 \\
\implies &\left(\sqrt{\log_x y} - \sqrt{\log_y x}\right)^2 = 0
\end{aligned}$$

Now note that if $t \in (0,1)$, then $\log t < 0$. Therefore as $x, y \in (0,1), \log_y x = \frac{\log x}{\log y} > 0$. Thus taking square roots is justified. Therefore

$$\log_x y - \log_y x = 0 \implies (\log x)^2 - (\log y)^2 = 0 \implies \log x - \log y = 0 \implies x = y$$

Since $\log x, \log y$ are both negative $\log x + \log y = 0$ was ruled out.

3. We prove by inducting on n. Label the unit squares as (x, y) where $x, y \in \{1, 2, \ldots, n\}$ and the top left corner is $(1,1)$. Consider the graph G_n with the n^2 unit squares as its vertices and two vertices are joined if and only if we can go form one point to the other via just one knight move. We want to prove that this G_n is connected.

   Base Case: $n = 4$. One can easily observe this by just looking at the graph for this case.

   Inductive Step. Suppose G_k is connected. One horse move gives us the following map for a vertex (x, y).

$$(x,y) \mapsto \{(x+2, y+1), (x+2, y-1), (x-2, y+1), (x-2, y-1)\}$$
$$(x,y) \mapsto \{(x+1, y+2), (x+1, y-2), (x-1, y+2), (x-1, y-2)\}.$$

   Note that G_n is connected if and only we can go from any point to $(1,1)$. By induction hypothesis, one can go to $(1,1)$ from any vertex $(x,y) \in S = \{(p,q) \mid p, q \in \{1, \ldots, k\}\}$. So we just need to take care of vertices of the form $(k+1, t)$ and $(t, k+1)$.

   For a vertex of the form $(k+1, t)$ where $t \in \{1, \ldots, k+1\}$, consider the map

$$(k+1, t) \mapsto (k, t \pm 2) \quad \text{depending on whether } t = 1, 2 \text{ or } k-1, k, k+1.$$

   For a vertex of the form $(t, k+1)$ where $t \in \{1, \ldots, k\}$, consider the map

$$(t, k+1) \mapsto (t \pm 2, k) \quad \text{depending on whether } t = 1, 2 \text{ or } k-1, k.$$

   Now all resulting vertices are in S, hence there is a path to $(1,1)$. Hence G_{k+1} is connected.

4. From the given equation we have

$$
\begin{aligned}
&\sum_{k=1}^{n} k\sqrt{x_k-k^2} = \frac{1}{2}\sum_{k=1}^{n} x_k \\
\Longrightarrow &\sum_{k=1}^{n}\left(x_k - 2k\sqrt{x_k-k^2}\right) = 0 \\
\Longrightarrow &\sum_{k=1}^{n}\left(\sqrt{x_k-k^2}-k\right)^2 = 0 \\
\Longrightarrow & x_k - k^2 = k^2 \forall k = 1,\ldots,n \\
\Longrightarrow & x_k = 2k^2 \forall k = 1,\ldots,n
\end{aligned}
$$

5. Let $a = a_1 < a_2 < \cdots < a_n = A$ be the n distinct positive integers. Suppose d and l are the gcd and lcm of these numbers respectively. Then $\frac{a_i}{d}$ are all positive integers. Also

$$1 \leq \frac{a_1}{d} < \frac{a_2}{d} < \cdots < \frac{a_n}{d}$$

Since $\frac{a_i}{d}$ is an integer strictly greater than $\frac{a_{i-1}}{d}$, we have $\frac{a_i}{d} \geq 1 + \frac{a_{i-1}}{d}$. Therefore

$$\frac{a_n}{d} \geq 1 + \frac{a_{n-1}}{d} \geq 1 + 1 + \frac{a_{n-2}}{d} \geq \cdots \geq n - 1 + \frac{a_1}{d} \geq n \Longrightarrow d \leq \frac{a_n}{n} = \frac{A}{n}$$

Similarly, we have that $\frac{l}{a_i}$ are all integers and

$$1 \leq \frac{l}{a_n} < \frac{l}{a_{n-1}} < \cdots < \frac{l}{a_1}$$

Since $\frac{l}{a_i}$ is an integer strictly greater than $\frac{l}{a_{i+1}}$, we have $\frac{l}{a_i} \geq 1 + \frac{l}{a_{i+1}}$. Therefore

$$\frac{l}{a_1} \geq 1 + \frac{l}{a_2} \geq 1 + 1 + \frac{l}{a_3} \geq \cdots \geq n - 1 + \frac{l}{a_n} \geq n \Longrightarrow d \geq na_1 = na$$

6. First observe that the points P_1 and P_2 can meet together only when both of them are on the ground surface. Suppose the wheels starts initially at point A on the surface and they meet again at point B on the surface t m away. Suppose the bigger wheel revolves k times to reach B. Therefore $12k = t$. So, 12 divides t. Similarly 8 must also divide t. Thus, t must be divisible by 24 . Since the points meet for the first time after A at B, value of t must be the minimum possible multiple of 24 , that is the required distance is 24 m.

Mathematics Talent Reward Programme

Question Paper for Senior Category

17^{th} January,2016

Total Marks: 150 Allotted Time: 10:00 a.m. to 12:30 p.m.

Multiple Choice Questions

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1. Sum of roots in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ of the equation $\sin x \tan x = x^2$ is

   a) $\frac{\pi}{2}$ b) 0 c) 1 d) None of these

2. Let f be a function satisfying $f(x+y+z) = f(x)+f(y)+f(z)$ for all integers x, y, z. Suppose $f(1) = 1$ and $f(2) = 2$. Then $\lim_{n\to\infty} \frac{1}{n^3} \sum_{r=1}^{n} 4rf(3r)$ equals

   a) 4 b) 6 c) 12 d) 24

3. z is a complex number and $|z| = 1$ and $z^2 \neq 1$. Then $\frac{z}{1-z^2}$ lies on

   a) a line not passing through origin b) $|z| = 2$
   c) x-axis d) y-axis

4. There are 168 primes below 1000 . Then sum of all primes below 1000 is

   a) 11555 b) 76127 c) 57298 d) 81722

5. ABCD is a quadrilateral on complex plane whose four vertices satisfy $z^4 + z^3 + z^2 + z + 1 = 0$. Then $ABCD$ is a

   a) Rectangle b) Rhombus c) Isosceles trapezium d) Square

6. Number of solutions of the equation $3^x + 4^x = 8^x$ in reals is

   a) 0 b) 1 c) 2 d) ∞

7. Let $\{x\}$ denote the fractional part of x. Then $\lim_{n\to\infty} \left\{(1+\sqrt{2})^{2n}\right\}$ equals

   a) 0 b) 0.5 c) 1 d) does not exist

8. Let p be a prime such that $16p + 1$ is a perfect cube. A possible choice for p is

   a) 283 b) 307 c) 593 d) 691

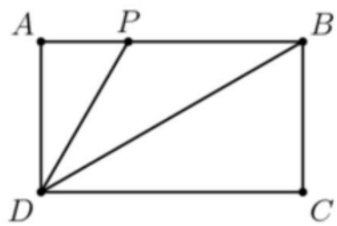
9. f be a function satisfying $2f(x) + 3f(-x) = x^2 + 5x$. Find $f(7)$.

   a) $-\frac{105}{4}$ b) $-\frac{126}{5}$, c) $-\frac{120}{7}$, d) $-\frac{132}{7}$.

10. Let $A = \{1, 2, \ldots, 100\}$. Let S be a subset of the power set of A such that any two elements of S has non zero intersection (Note that elements of S are actually some subsets of A). Then the maximum possible cardinality of S is

    a) 2^{99} b) $2^{99} + 1$ c) $2^{99} + 2^{98}$ d) None of these

11. In rectangle $ABCD, AD = 1, P$ is on $\overline{AB}$, and $\overline{DB}$ and $\overline{DP}$ trisect $\angle ADC$. What is the perimeter of



a) $3+\frac{\sqrt{3}}{3}$, b) $2+\frac{4\sqrt{3}}{3}$, c) $2+2\sqrt{2}$, d) $\frac{3+3\sqrt{5}}{2}$.

12. Let $f(x)=(x-1)(x-2)(x-3)$. Consider $g(x)=\min\{f(x),f'(x)\}$. Then the number of points of discontinuity are

a) 0 b) 1 c) 2 d) more than 2

13. Let $P(x)=x^2+bx+c$. Suppose $P(P(1))=P(P(-2))=0$ and $P(1)\neq P(-2)$. Then $P(0)=$

a) $-\frac{5}{2}$ b) $-\frac{3}{2}$ c) $-\frac{7}{4}$ d) $\frac{6}{7}$

14. Let $[x]$ denotes the greatest integer less than or equal to x. Find x such that $x[x[x[x]]]=88$

a) π b) 3.14 c) $\frac{22}{7}$ d) All of these

15. Suppose $50x$ is divisible by 100 and kx is not divisible by 100 for all $k=1,2,\ldots,49$. Find number of solutions for x when x takes values $1,2,\ldots,100$.

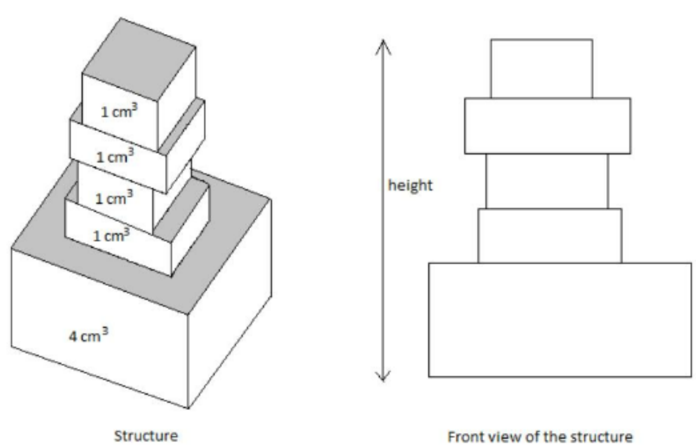
a) 20 b) 25 c) 15 d) 50

Short Answer Type Questions

[Each question carries a total of 15 marks. Credits will be given to partially correct answers]

1. Show that there exist a polynomial $P(x)$ whose one coefficient is $\frac{1}{2016}$ and remaining coefficients are rational numbers, such that $P(x)$ is an integer for any integer x.

2. 5 blocks of volume $1\text{ cm}^3, 1\text{ cm}^3, 1\text{ cm}^3, 1\text{ cm}^3$, and 4 cm^3 are placed one above another to form the structure as shown in the figure. Suppose the sum of surface areas of upper face of each block is 48 cm^2. Determine the minimum possible height of the whole structure.



3. Prove that for any positive integer n there are n consecutive composite numbers all less than 4^{n+2}. [You may use the fact that product of all primes, which are less than k, is less than 4^k and this holds for all positive integers k.]

4. For any given k points in a plane, we define the diameter of the points as the maximum distance between any two points among the given points. Suppose n point are there in a plane with diameter d. Show that we can always find a circle with radius $\frac{\sqrt{3}}{2}d$ such that all the points lie inside the circle.

5. Let $\mathbb{N}$ be the set of all positive integers. $f, g : \mathbb{N} \to \mathbb{N}$ be functions such that f is onto, g is one-one and $f(n) \geq g(n)$ for all positive integers n. Prove that $f = g$.

6. Consider the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. A partition Π of A is a collection of disjoint sets whose union is A. For example, $\Pi_1 = \{\{1, 2\}, \{3, 4, 5\}, \{6, 7, 8, 9\}\}$ and $\Pi_2 = \{\{1\}, \{2, 5\}, \{3, 7\}, \{4, 6, 8, 9\}\}$ can be considered as partitions of A. For each Π partition, we consider the function π defined on the elements of A. $\pi(x)$ denotes the cardinality of the subset in Π which contains x. For example, in case $\Pi_1, \pi_1(1) = \pi_1(2) = 2, \pi_1(3) = \pi_1(4) = \pi_1(5) = 3$, and $\pi_1(6) = \pi_1(7) = \pi_1(8) = \pi_1(9) = 4$. For Π_2 we have, $\pi_2(1) = 1$, $\pi_2(2) = \pi_2(5) = 2, \pi_2(3) = \pi_2(7) = 2$, and $\pi_2(4) = \pi_2(6) = \pi_2(8) = \pi_2(9) = 4$. Given any two partitions Π and Π', show that there are two numbers x and y in A, such that $\pi(x) = \pi'(x)$ and $\pi(y) = \pi'(y)$. [Hint: Consider the case where there is a block of size greater than or equal to 4 in a partition and the alternative case.]

Use of calculators is not allowed. You may use a ruler and a compass for construction.

∼ Best of Luck ∼

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Model Solutions for Senior Category

Multiple Choice Questions

[Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. (B) [Observe that if a is a solution, then $-a$ is a solution]
2. (A) [$f(n+2) = f(n) + 2$, by induction $f(n) = n$]
3. (D) [$\frac{z(1-\bar{(z)}}{|1-z^2|^2}$]
4. (B) [There are 167 odd primes, hence sum is odd, moreover $p_n > n$,i.e. sum¿$\frac{167.168}{2}$]
5. (C)[Any 4 vertices of a regular pentagon makes an isosceles trapezium]
6. (B) [Take $f(x) = (\frac{3}{8})^x + (\frac{4}{8})^x$, hence $f(0) > 1, f(1) < 1$, and f is monotonically decreasing.]
7. (C) [observe $a_n = (\sqrt{2}-1)^n+(\sqrt{2}+1)^n$ is an integer]// 8. (B) [let $16p+1 = n^3$, i.e. $16p = (n-1)(n^2+n+1)$. Now $n^2 + n + 1$ is odd, hence equals 1 or p. Also $n - 1 < n^2 + n + 1$,hence n-1=16]
9. (B) [replace x with -x and solve for f(x)]
10. (A) [No. of subsets consisting 1 is 2^{99}, hence $|S| \geq 2^{99}$. Now consider the set of subsets which contain 1, and set of there compliments. These are disjoint and has cardinality $2^{99} + 1$. Now, if a set of $2^{99} + 1$ subset had that property, then by PHP, it would consisted (X,X^c).]
11. (B)
12. (D) [sketch the graphs]
13. (A) [P(1) and P(-2) are two roots of the quadratic, now observe the sum of roots]
14. (C) [Observe that x is rational]
15. (B) [x is even but not divisible by 5]

Short Answer Type Questions

[Each question carries a total of 15 marks. Credit will be given to partially correct answers]

1. Consider $P(x) = \frac{1}{2016}(x-1)(x-2)\cdots(x-2016)$. Clearly all coefficients of $P(x)$ are rationals. Observe that the leading coefficient of $P(x)$ is $\frac{1}{2016}$. Note that product of 2016 consecutive integers is always divisible by 2016 . Hence this $P(x)$ is our required polynomial.

2. First Solution: Let $a_1, a_2, \ldots, a_5$ be the surface areas of upper face of the blocks. We have $a_1 + a_2 + \cdots + a_5 = 48$. Note that the heights of the blocks are given by

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}, \text{ and } \frac{4}{a_5}$$

Note that by Cauchy-Schwarz inequality

$$\begin{aligned}&48\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{4}{a_5}\right)\\&=\left(\frac{1}{(\sqrt{a_1})^2}+\frac{1}{(\sqrt{a_2})^2}+\frac{1}{(\sqrt{a_3})^2}+\frac{1}{(\sqrt{a_4})^2}+\left(\frac{2}{\sqrt{a_5}}\right)^2\right)\left((\sqrt{a_1})^2+\cdots+(\sqrt{a_5})^2\right)\\&\geq(1+1+1+1+2)^2=36\end{aligned}$$

Hence

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{4}{a_5}\geq\frac{36}{48}=\frac{3}{4}$$

Equality holds when $a_5 = 2a_1 = 2a_2 = 2a_3 = 2a_4 = 16$. Thus minimum possible height is certainly $\frac{3}{4}$.
Second solution: One can arrive at the same conclusion using AM-HM inequality. Observe that

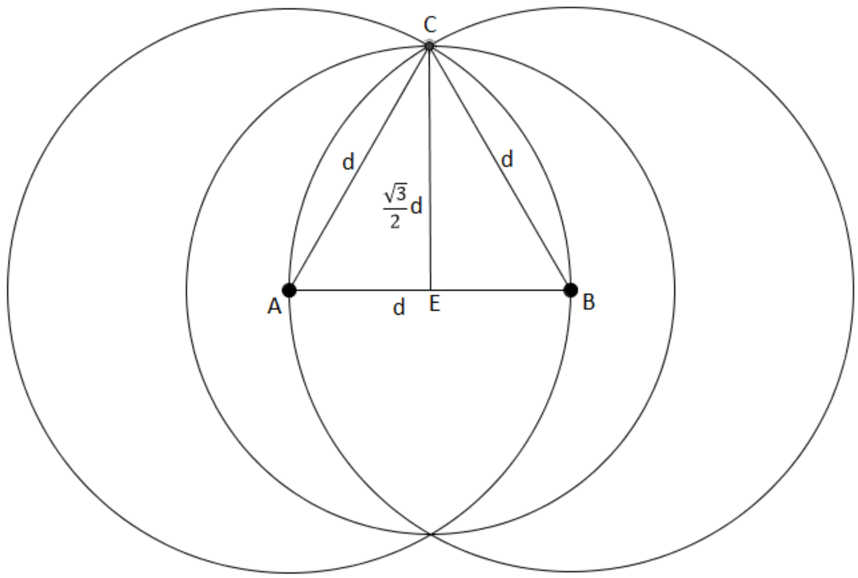
$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{4}{a_5}$$
$$=\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{1}{a_5/2}+\frac{1}{a_5/2}$$
$$\geq \frac{36}{a_1+a_2+a_3+a_4+a_5/2+a_5/2}=\frac{36}{48}=\frac{3}{4}$$

3. Let P be the product of all primes which are less than or equal to $n+1$. Consider the following n consecutive numbers

$$P+2, P+3, \ldots, P+(n+1)$$

We will now show that $P+k$ is composite for any $k \in \{2,3,\ldots n+1\}$. Consider any prime divisor of k say u. Clearly u divides P. Thus u divides $P+k$ implying $P+k$ is composite. This completes the proof.

4. Consider the two points say A and B whose distance is d. Now all points must lie within the circle centre at A as well as the circle centre at B. So they must lie within the intersection of the two circles. Consider the circle drawn with centre as the mid point of A and B and with radius the altitude of the shown equilateral triangle ABC. Note that the radius equals $\frac{\sqrt{3}}{2}d$. This new circle covers the intersection region of the two circles and hence all the points within it.



5. We shall prove the statement $P(k)$: For every $k \in \mathbb{N}$ there exists a unique $x_k \in \mathbb{N}$ such that $f(x_k) = g(x_k) = k$ by induction on k. Since f is onto, there exists $x_1 \in \mathbb{N}$ such that $f(x_1) = 1$. But $g \leq f$ so $g(x_1) = 1$. Since g is one-one this x_1 is unique. Thus we have proved $P(1)$. Now let $P(k)$ be true. We shall prove that $P(k+1)$ is true. As f is onto, there exists $x_{k+1} \in \mathbb{N}$ such that $f(x_{k+1}) = k+1$. But $g \leq f$ and by induction hypothesis g already takes all values less that $k+1$. So $g(x_{k+1}) = k+1$. Since g is one-one this x_{k+1} is unique. Thus by the principle of mathematical induction, the statement $P(k)$ holds for all natural numbers k. Observe that $P(k)$ implies $f = g$. This completes the proof.

6. Part 1: Let us assume for the sake of contradiction there do not exist two distinct elements satisfying the property.

Part 2: If there were 4 distinct values of π or π' in decreasing, the first value is at least 4 , the second value is at least 3 , the third value is at least 2 , and the fourth value is at least 1 , so we need at least 4 elements in the first partition, 3 in the second partition, 2 in the third partition, and 1 in the fourth partition, which implies we need at least 10 elements. But $4+3+2+1>9$. Hence there can only be at most 3 distinct values for $\pi(x)$ and $\pi'(x)$.

Part 3: In addition, only at most three elements can share a value for $\pi(x)$ or $\pi'(x)$ If there were 4 elements which had the same value for $\pi(x)$, WLOG $\pi(1)=\pi(2)=\pi(3)=\pi(4)$, then by Part 1 we know $\pi'(1), \pi'(2), \pi'(3), \pi'(4)$ must all be different, but this contradicts Part 2 . So for each value of $\pi(x)$, only at most three elements can share that value.

So we know from Part 3 that $\pi(x)$ cannot equal 4 . It follows that the only possible way is for 3 elements to satisfy $\pi(x)=3,3$ other elements to satisfy $\pi(x)=2$, and 3 other elements to satisfy $\pi(x)=1$. But it's impossible for exactly three elements to satisfy $\pi(x)=2$, because the number of elements satisfying $\pi(x)=2$ must be even. So we have a contradiction.

Mathematics Talent Reward Programme

Question Paper for Senior Category

15^{th} January,2017

Total Marks: 102 Allotted Time: 10:00 a.m. to 12:30 p.m.

Multiple Choice Questions

[You should answer these questions in the first page according to the order given in the question. Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. The number of real solutions of the equation $(9/10)^x = -3 + x - x^2$ is

   a) 2 b) 0 c) 1 d) None of these

2. $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} =$

   a) $\sqrt{e}$, b) ∞ c) Does not exist d) None of these

3. Let $p(x) = x^4 - 4x^3 + 2x^2 + ax + b$. Suppose that for every root λ of $p, 1/\lambda$ is also a root of p. Then $a + b =$

   a) -3 b) -6 c) -4 d) -8

4. Let $F_1 = F_2 = 1$. We define inductively $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 2$. Then the sum

   $$F_1 + F_2 + F_3 + \cdots + F_{2017}$$

   is

   a) even but not divisible by 3 b) odd but divisible by 3
   c) odd and leaves remainder 1 when divided by 3 d) NOne of these

5. Compute the number of ordered quadruples of positive integers (a, b, c, d) such that

   $$a! \cdot b! \cdot c! \cdot d! = 24!$$

   a) 4 b) 6 c) 4^4 d) None of these

6. Let $p(x)$ is a polynomial of degree 4 with leading coefficients 1 . Suppose $p(1) = 1, p(2) = 2, p(3) = 3$ and $p(4) = 4$. Then $p(5) =$

   a) 5 b) $\frac{25}{6}$ c) 29 d) 35

7. Let $ABCD$ be a quadrilateral with sides $AB = 2, BC = CD = 4$ and $DA = 5$. The opposite angles A and C are equal. The length of diagonal BD equals

   a) (A) $2\sqrt{6}$, b) (B) $3\sqrt{3}$, c) (C) $3\sqrt{6}$, d) (D) $2\sqrt{3}$.

8. How many finite sequences $x_1, x_2, \ldots, x_m$ are there such that $x_i = 1$ or 2 and $\sum_{i=1}^{m} x_i = 10$?

   a) 89 b) 73 c) 107 d) 119

9. From a point P outside of a circle with centre O, tangent segments PA and PB are drawn. If $\frac{1}{OA^2} + \frac{1}{PA^2} = \frac{1}{16}$. Then $AB =$

a) 4 b) 6 c) 8 d) 10

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $\lim_{x\to\infty} f'(x) = 1$, then

    a) f is increasing, b) f is unbounded, c) f' is bounded, d) All of these

Short Answer Type Questions

[Each question carries a total of 12 marks. Credits will be given to partially correct answers]

1. A monic polynomial is a polynomial whose highest degree coefficient is 1 . Let $P(x)$ and $Q(x)$ be monic polynomials with real coefficients, and $\deg P(x) = \deg Q(x) = 10$. Prove that if the equation $P(x) = Q(x)$ has no real solutions, then $P(x+1) = Q(x-1)$ has a real solution.

2. Let a, b, c be positive reals such that $a+b+c = 3$. Show that

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \leq \frac{6}{\sqrt{(a+b)(b+c)(c+a)}}$$

3. Let $f : [0,1] \to [0,1]$ be a continuous function. We say $f \equiv 0$ if $f(x) = 0$ for all $x \in [0,1]$ and similarly $f \not\equiv 0$ if there exist at least one $x \in [0,1]$ such that $f(x) \neq 0$. Suppose $f \not\equiv 0, f \circ f \not\equiv 0$, but $f \circ f \circ f \equiv 0$. Do there exist such an f ? If yes construct such an function, if no prove it. [Note that $f \circ f(x) = f(f(x))$ and $f \circ f \circ f(x) = f(f(f(x)))$.]

4. 4. An irreducible polynomial is a non-constant polynomial that cannot be factored into the product of two non-constant polynomials. Consider the following statements:

   Statement 1: $p(x)$ be any monic irreducible polynomial with integer coefficients and degree ≥ 4. Then $p(n)$ is prime for at least one natural number n.

   Statement 2: $n^2 + 1$ is prime for infinitely many values of natural number n.

   Show that if Statement 1 is true then Statement 2 is also true.

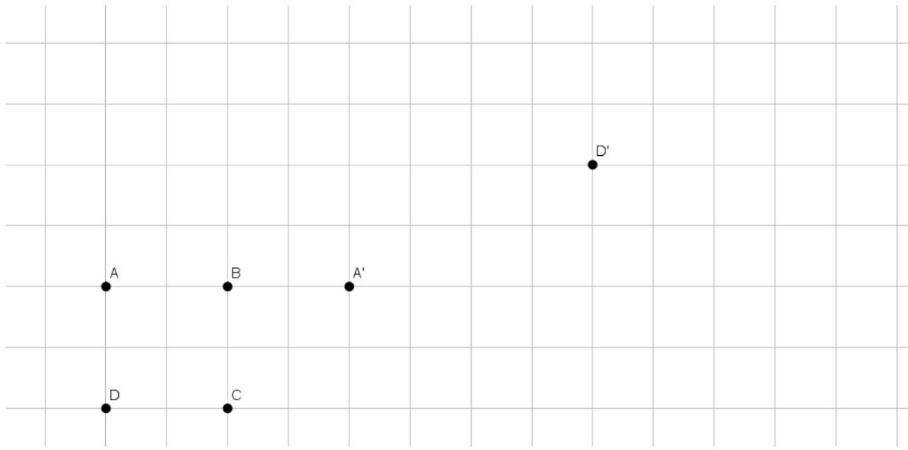
5. Let $\mathbb{N}$ be the set of all natural numbers. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijective function. Show that there exist three numbers a, b, c in arithmetic progression such that $f(a) < f(b) < f(c)$.

6. Let us consider an infinite grid plane as shown below. We start with 4 points A, B, C, D, that form a square, as shown below.

   We perform the following operation: We pick two points say X and Y from the current points. X is reflected about Y to get X'. We remove X and add X' to get a new set of 4 points and treat it as our current points.

   For example in the figure suppose we choose A and B (we can choose any other pair too). Then reflect A about B to get A'. We remove A and add A' Thus A', B, C, D is our new 4 points. We may again choose D and A' from the current points. Reflect D about A to obtain D' and hence A', B, C, D' are now new set of points. Then similar operation is performed on this new 4 points and so on.

   Starting with A, B, C, D, can you get a bigger square by some sequence of such operations?



Use of calculators is not allowed. You may use a ruler and a compass for construction.

∼ Best of Luck ∼

Mathematics Talent Reward Programme

Model Solutions for Senior Category

Multiple Choice Questions

[Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. (B)[Find the signs of RHS and LHS]
2. (D) [Note the limit is of the form 1^∞, $\lim_{x\to 0}(\frac{sinx}{x})^{\frac{1}{x^2}} = e^{\lim_{x\to 0}\frac{1}{x^2}\cdot(\frac{sinx}{x}-1)}$]
3. (A)[Note in this case sum of the roots = sum of the roots taken 3 at a time]
4. (C)[Fibonacci numbers are periodic modulo 2 with period $1,1,0$ and they are also periodic modulo 3 with period $1,1,2,0,2,2,1,0$]
5. (D)[Note 23 is a prime] 6. (C)[Consider the roots of $h(x) = p(x) - x$]
7. (A)[Recall the cosine rule]
8. (A)[Let there are a terms with $x_i = 1$ and b terms with $x_i = 2$]
9. (C)[Use Pythagoras theorem and equate area of triangle OAP]
10. (B)

Short Answer Type Questions

[Each question carries a total of 12 marks. Credit will be given to partially correct answers]

1. Let $P(x) = x^{10} + a_9x^9 + a_8x^8 + \cdots + a_0$ and $Q(x) = x^{10} + b_9x^9 + b_8x^8 + \cdots + b_0$. Let $R(x) = P(x) - Q(x)$ Note that the equation $R(x) = (a_9 - b_9)x^9 + (a_8 - b_8)x^8 + \cdots + (a_0 - b_0)$ If $a_9 \neq b_9$, then $R(x)$ is of degree 9 , then the polynomial $R(x)$ must have a real root which contradicts the assumption that $R(x) = P(x) - Q(x) = 0$ has no real solutions. Thus $a_9 = b_9$.

   Let $S(x) = P(x+1) - Q(x-1)$. Then

   $$S(x) = (x+1)^{10} - (x-1)^{10} + a_9(x+1)^9 - a_9(x-1)^9 + T(x)$$

   where $T(x)$ is polynomial of degree at most 8 . Clearly $a_9\left[(x+1)^9 - (x-1)^9\right]$ is of degree atmost 8 since on expansion x^9 coefficient cancels out, whereas

   $$\begin{aligned}(x+1)^{10} - (x-1)^{10} &= \left[x^{10} + 10x^9 + A(x)\right] - \left[x^{10} - 10x^9 + B(x)\right] \\ &= 20x^9 + A(x) - B(x)\end{aligned}$$

   where $A(x)$ and $B(x)$ are polynomials of degree at most 8 . Hence $S(x)$ is of degree exactly equal to 9 hence it must have a real root. Thus $P(x+1) - Q(x-1)$ has real solution.

2. Note that the given inequality can be written as

   $$\sqrt{a(a+b)(a+c)} + \sqrt{b(b+c)(b+a)} + \sqrt{c(c+a)(c+b)} \leq 6$$

   Note that

   $$\begin{aligned}(a+b+c)^2 - 3(ab+bc+ca) &= a^2 + b^2 + c^2 - ab - bc - ca \\ &= \frac{1}{2}(a-b)^2 + \frac{1}{2}(b-c)^2 + \frac{1}{2}(c-a)^2 \geq 0\end{aligned}$$

   Hence $(a+b+c)^2 \geq 3(ab+bc+ca)$. Since $a+b+c = 3$, we have that $ab+bc+ca \leq 3$.

   Solution 1:

By Cauchy Schwarz inequality we have that

$$(a+b+c)((3a+bc)+(3b+ca)+(3c+ab)) \geq (\sqrt{a(3a+bc)}+\sqrt{b(3b+ca)}+\sqrt{c(3c+ab)})^2$$

Note that $a+b+c=3$ and hence $3a+bc=(a+b+c)a+bc=(a+b)(a+c), 3b+ca=(b+c)(b+a)$, and $3c+ab=(c+a)(c+b)$. Thus taking square roots in the above inequality we have

$$\sqrt{3(3a+3b+3c+ab+bc+ca)} \geq \sqrt{a(a+b)(a+c)}+\sqrt{b(b+c)(b+a)}+\sqrt{c(c+a)(c+b)}$$

Note that $ab+bc+ca \leq 3$, hence

$$\sqrt{a(a+b)(a+c)}+\sqrt{b(b+c)(b+a)}+\sqrt{c(c+a)(c+b)} \leq \sqrt{3(3\times 3+3)} = \sqrt{36} = 6$$

Solution 2: Observe that by AM-GM inequality we have

$$\frac{7a+bc}{2} = \frac{4a+(3a+bc)}{2} \geq \sqrt{4a(3a+bc)}$$

Since $a+b+c=3, 3a+bc=(a+b+c)a+bc=(a+b)(a+c)$. Hence $\sqrt{4a(a+b)(a+c)} \leq \frac{1}{2}(7a+bc)$. We divide both sides by 2 to obtain

$$\sqrt{a(a+b)(a+c)} \leq \frac{1}{4}(7a+bc)$$

Analogously we obtain

$$\sqrt{b(b+c)(b+a)} \leq \frac{1}{4}(7b+ca)$$
$$\sqrt{c(c+a)(c+b)} \leq \frac{1}{4}(7c+ab)$$

Adding all three inequalities we have

$$\begin{aligned}
&\sqrt{a(a+b)(a+c)}+\sqrt{b(b+c)(b+a)}+\sqrt{c(c+a)(c+b)} \\
&\leq \frac{1}{4}(7a+bc)+\frac{1}{4}(7b+ca)+\frac{1}{4}(7c+ab) \\
&= \frac{1}{4}(7(a+b+c)+(ab+bc+ca)) \\
&\leq \frac{1}{4}(7\times 3+3) = 6
\end{aligned}$$

The last inequality is due to the fact that $a+b+c=3$ and $ab+bc+ca \leq 3$.

3. There exists such a function satisfying all conditions. We construct one such function.

   We first show how to get an f such that $f \not\equiv 0$ but $f \circ f \equiv 0$ Let $A = \{x \in [0,1] \mid f(x) = 0\}$ and $B = \{x \in [0,1] \mid f(x) \neq 0\}$ be the set where f takes value non zero. Since $f(f(x))$ is zero for all $x, f(x)$ must take values in A. If we take $A = [0, 1/2]$, we have to ensure that f is continuous, $f(x) > 0$ for all $x > 1/2$ and $f(x) \leq \frac{1}{2}$ for all x. To do this we may take f as a part of a line whose slope is sufficiently small so that $f(x) \leq \frac{1}{2}$ for all x. For example we may take $f(x) = \left(x - \frac{1}{2}\right)$ for $x \geq \frac{1}{2}$. Note that continuity is maintained and $f \not\equiv 0$ and $f(x) \leq \frac{1}{2}$ for all x. Hence $f \circ f \equiv 0$.

   To do the part where $f \not\equiv 0, f \circ f \not\equiv 0$ but $f \circ f \circ f \equiv 0$, we apply the same idea, however the time we increase the slope so that $f \circ f \not\equiv 0$. Suppose $f(x) = 0$ for $x \leq 1/2$ and $f(x) = k\left(x - \frac{1}{2}\right)$

for $x \geq 1/2$ where $1 < k < 2$. Then $f(f(1)) = f(k/2) \neq 0$ as $\frac{k}{2} \geq \frac{1}{2}$. So $f \circ f \not\equiv 0$. To ensure $f \circ f \circ f \equiv 0$, we note that f is a non decreasing function hence it attains maximum at $x = 1$. But $f(f(1)) = f(k/2) = \frac{1}{2}k(k-1)$ which is less than $\frac{1}{2}$ as long as we choose sufficiently close to 1. We may choose $k = \frac{3}{2}$ for example for this purpose. Then $f(f(1)) = \frac{3}{8} < \frac{1}{2}$. Then $f \circ f$ is always less than $\frac{1}{2}$ which forces $f \circ f \circ f \equiv 0$. So the following functions works:

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1/2] \\ \frac{3}{2}\left(x - \frac{1}{2}\right) & \text{if } x \in (1/2, 1] \end{cases}$$

4. Suppose statement 1 is true but statement 2 is false. Hence $n^2 + 1$ is prime for finitely many values of n. Hence there exist $N > 0$ such that $n^2 + 1$ is composite for all $n > N$. Let $p(x) = x^4 + 1$. It is easy to check that $p(x)$ is irreducible. Note that this implies $p(x + k)$ is irreducible for any k. Finally we consider $p(x + N) = (x + N)^4 + 1$. Note that statement 2 implies $p(n + N)$ is prime at least for one natural number n. Hence suppose $p(n_0 + N)$ is prime for some natural number n_0, But $p(n_0 + N) = \left((n_0 + N)^2\right)^2 + 1$ and $(n_0 + N)^2 > N^2 \geq N$ which is a contradiction to the fact that $n^2 + 1$ is always composite after $n > N$.

5. Since f is bijection let us choose a such that $f(a) = 1$. If $f(a) < f(a+1) < f(a+2)$ we are done. If not the only other possibility is $f(a) < f(a+2) < f(a+1)$. This implies $f(a+2)$ lies between $f(a+1)$ and $f(a)$. We then consider $a, a+2, a+4$. If $f(a) < f(a+2) < f(a+4)$ we are done. If not the only other possibility is $f(a) < f(a+4) < f(a+2)$. Since $f(a+2) < f(a+1), f(a+4)$ lies between $f(a+1)$ and $f(a)$. We then consider $a, a+4, a+8$. If $f(a) < f(a+4) < f(a+8)$ does not hold, we can again conclude that $f(a+8)$ lies between $f(a+1)$ and $f(a)$ and so on.

   Note that there are finitely many natural numbers between $f(a+1)$ and 1 . Since f is a bijection, only for finitely many values of $n, f(n)$ lies between $f(a+1)$ and 1 . So if we continue in the above fashion we must get a n_0 such that $f(a) < f\left(a + 2^{n_0}\right) < f\left(a + 2^{n_0+1}\right)$.

6. Let $ABCD$ be the initial square. Suppose it is possible to reach a bigger square say $EFGH$. Note that it is not necessary that sides of $EFGH$ is parallel to the grid lines.

   We claim that the operations are reversible i.e., if starting from P, Q, R, S you reach P', Q', R', S' then you can come back to P, Q, R, S by some sequence of operations. To prove this, consider two points X and Y. Suppose we reflect X about Y to get X'. Then according to the rule we remove X and add X'. We can now reflect X' about Y to get back X. We may remove X' and add X to get back the two points. Thus such an operation is reversible. Clearly for a sequence of operations, we may apply the reversibility of each operations to get the reversed sequence of operations. This proves our claim.

   So there is a sequence of operations by which starting from $EFGH$ one can reach smaller square $ABCD$. Now without loss of generality assume $E = (0, 0), F = (0, 1), G = (1, 1)$ and $H = (1, 0)$.

   We now claim that every point that can arise by this operation has integer co-ordinates. This is true because if $X = (x_1, x_2)$ and $Y = (y_1, y_2)$, then $X' = (2y_1 - x_1, 2y_2 - x_2)$. So inductively every such point must have integer co-ordinates.

   $ABCD$ is smaller square then $EFGH$. According to the co-ordinate system that we impose, $EFGH$ has side length 1. So, $ABCD$ has side length less than 1. But A, B, C, D must have all integer co-ordinates. Since any two distinct points having integer co-ordinates is at least 1 distance apart, this gives us a contradiction.

Mathematics Talent Reward Programme

Question Paper for Senior Category

14^{th} January,2018

Total Marks: 100 Allotted Time: 10:00 a.m. to 12:30 p.m..

Multiple Choice Questions

[You should answer these questions in the first page according to the order given in the question. Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. A coin is tossed 9 times. There are 2^9 possible outcomes. In how many of these outcomes does no two successive heads occur?

   a) 55 b) 34 c) 89 d) None of these

2. $\lim\limits_{x\to 0^+} \dfrac{[x]}{\tan(x)} =$

   a) -1 b) 1 c) 0 d) does not exist

3. Let F_n denote the Fibonacci sequence such that $F_1 = 0, F_2 = 1, F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 3$. Then $\displaystyle\sum_{n=3}^{\infty} \frac{18 + 999F_n}{F_{n-1}F_{n+1}} =$

   a) 2016 b) 2017 c) 2018 d) None of these

4. In $\triangle ABC, O$ is an interior point such that $\angle BOC = 90°, \angle CAO = \angle ABO, \angle BAO = \angle BCO$. Then $\frac{AC}{OC} =$

   a) $\sqrt{2}$ b) 2 c) $\sqrt{\frac{3}{2}}$ d) None of these

5. Let M and m denote the maximum value and the minimum value of the function $f(x) = \cos\left(x^{2018}\right)\sin(x)$ in the interval $[-2\pi, 2\pi]$ respectively, then $m + M =$

   a) $\frac{1}{2}$ b) $-\frac{1}{\sqrt{3}}$ c) $\frac{1}{2018}$ d) None of these

6. In a class of 80 students, 40 are male and 40 are female. Also, exactly 50 students wear glasses. Then which of the following is true?

   a) Exactly 10 boys wear glasses b) At least 20 girls wear glasses

   c) At most 25 boys do not wear glasses d) At most 30 girls do not wear glasses

7. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. How many functions $f : A \to A$ can be defined such that $f(1) < f(2) < f(3)$?

   a) $\begin{pmatrix} 8 \\ 3 \end{pmatrix}$, b) $\begin{pmatrix} 8 \\ 3 \end{pmatrix} 5^8$ c) $\begin{pmatrix} 8 \\ 3 \end{pmatrix} 8^5$ d) $\frac{8!}{3!}$.

Short Answer Type Questions

[Each question carries a total of 12 marks. Credits will be given to partially correct answers]

1. If x, y, z are real numbers such that $x < y < z$, prove that

$$(x - y)^3 + (y - z)^3 + (z - x)^3 > 0$$

2. Let $P(x)$ be a polynomial with real coefficients such that $P(n)$ is an integer for any integer n. Prove that the coefficients of $P(x)$ must be rational.

3. Does there exist a continuous function f, such that $f(f(x)) = -x^{2019} \quad \forall x \in \mathbb{R}$?

4. Let S be a finite subset of $\mathbb{R}$. Let f be a function from S to S such that $|f(x_1) - f(x_2)| \leq \frac{1}{2}|x_1 - x_2|$ $\forall x_1, x_2 \in S$. Prove that $f(x) = x$ for some $x \in S$.

5. (a) Prove that the sequence of remainders obtained when the Fibonacci numbers are divided by n is periodic, where $n \in \mathbb{N}$.
(b) Prove that there does not exist a non-constant polynomial $P(x)$ with integer coefficients such that $P(F_n)$ is prime for all $n \in \mathbb{N}$, where F_n denotes the nth term of the Fibonacci sequence.

6. Let $d(n)$ be the number of divisors of n. Prove that we can colour the natural numbers using 2 colours such that if for an infinite increasing sequence $\{a_1, a_2, a_3, \cdots\}$, the sequence $\{d(a_1), d(a_2), \cdots\}$ is a non-constant geometric progression, then all the terms $\{a_1, a_2, a_3 \cdots\}$ cannot have the same colour. (You may use the fact that we can colour the natural numbers using 2 colours such that all the terms of any infinite increasing A.P cannot have the same colour.)

Use of calculators is not allowed. You may use a ruler and a compass for construction.

∼ Best of Luck ∼

Mathematics Talent Reward Programme

Model Solutions for Senior Category

Multiple Choice Questions

[Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. (C)[Find the no of ways to choose k non-consecutive places out of 9 places and put H in those selected places]
2. (C) $\left[\dfrac{\lfloor x\rfloor}{\tan x} = 0 \forall x \in (0,1)\right]$
3. (A)$\left[\frac{1}{F_{n-1}F_{n+1}} = \frac{1}{F_{n-1}F_n} - \frac{1}{F_nF_{n+1}}\right]$ and $\left[\frac{F_n}{F_{n-1}F_{n+1}} = \frac{1}{F_{n-1}} - \frac{1}{F_{n+1}}\right]$
4. (A)[Extend OC to C' such that O is midpoint of CC'.Conclude that $AOBC'$ is cyclic. Now, $\triangle C'AC \sim \triangle AOC$]
5. (D)[$f(x)$ is an odd function]
6. (D)
7. (B)[First choose 3 no's out of 8 for $f(1), f(2), f(3)$]

Short Answer Type Questions

[Each question carries a total of 12 marks. Credit will be given to partially correct answers]

1. Call $y - x = a > 0$, $z - y = b > 0$. Then $z - x = a + b$. We have to show that

$$-a^3 - b^3 + (a+b)^3 = 3ab(a+b) > 0$$

which is trivially true since both a and b are strictly positive.

2. We have $P(i) = a_i$ for $i = 1, 2, \ldots d+1$, where d is the degree of the polynomial, and $a_1, a_2, \ldots a_{d+1}$ are integers. Hence, by Lagrange's interpolation, we may obtain a d degree polynomial Q with rational coefficients s.t. $Q(i) = a_i$ for $i = 1, 2, \ldots d+1$. As P and Q are both d degree polynomials, $P - Q$ is at most a d degree polynomial and has roots $1, 2, \ldots d+1$, i.e. d+1 roots, and so is identically 0. Hence $P \equiv Q$. Since Q has rational coefficients, we conclude that P has rational coefficients as well.

3. Let, if possible, there exists a continuos function f such that

$$f(f(x)) = -x^{2019} \qquad \forall x \in \mathbb{R}$$

Clearly f is bijective (and hence one-one) and continuos which implies that f is monotone.

Case 1: Assume that f is monotonically increasing. Therefore, we have that

$$x > y \implies f(x) > f(y) \implies f(f(x)) > f(f(y)) \implies -x^{2019} > -y^{2019}$$

which is absurd.

Case 2: Assume that f is monotonically decreasing. Therefore, we have that

$$x > y \implies f(y) > f(x) \implies f(f(x)) > f(f(y)) \implies -x^{2019} > -y^{2019}$$

which is again absurd.

Hence, we conclude that no such f can exist.

4. Let $\{s_i : 1 \le i \le n\}$ be an enumeration of S. The set S has n elements. Let,

$$d = \min\{|s_i - s_j| : 1 \le i < j \le n\}$$

.

Now, for distinct $x, y \in S$,

$$|f^n(x) - f^n(y)| \leq \frac{1}{2}|f^{n-1}(x) - f^{n-1}(y)| \leq \ldots \frac{1}{2^n}|x - y|$$

So, for large enough n,

$$\frac{1}{2^n}|x - y| < d$$

$\forall\ x, y \in S$.

Then, for this n, $\forall x, y \in S$,

$$f^n(x) = f^n(y) = k$$

.

We show that k is a fixed point.

$$k = f^n(f(x)) = f(f^n(x)) = f(k)$$

So, k is a fixed point.

5. **(a)** For two consecutive remainders in the sequence, we have n^2 choices(Set of remainders is $\{0, 1, 2, \ldots, n-1\}$ upon division by n).

   Since the Fibonacci sequence has infinitely many terms , we shall have some pair of consecutive remainders repeat. Let $F_{k_1} \equiv F_{k_2} \pmod{n}$ and $F_{k_1+1} \equiv F_{k_2+1} \pmod{n}$ for some $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$.

   Verify that $F_n \equiv F_{n+(k_2-k_1)}\ \forall n \in \mathbb{N}$ using the property $F_{n+2} = F_{n+1} + F_n$.

   (b) Let $F_n = p$ for some $n \in \mathbb{N}$. From (a), it may be concluded that, $\exists k \in \mathbb{N}$ such that $\forall m \in \mathbb{N}$,

$$F_{m+k} \equiv F_m \pmod{p}$$

   .

   Now, for $a \in \mathbb{N}$, $F_{an} \equiv F_n \pmod{p}$ and so,

$$P(F_{an}) \equiv P(F_n) \pmod{p}$$

   Since $P(F_{an})$ is a prime, $P(F_{an}) = p\ \forall a \in \mathbb{N}$. Implying that the polynomial given by, $Q(x) = P(x) - p$ has infinitely many distinct roots and hence, is identically zero. Therefore, P is a constant polynomial, a contradiction.

6. We construct one such colouring.

   Each natural number has at least two divisors, that is, 1 and the number itself. So, we colour the numbers green for which $2^{(2k-1)^2} < d(n) \leq 2^{(2k)^2}$ and color the rest red.

   Let r be the common ratio of the geometric progression. Let $n \in \mathbb{N}$ be such that $2^n > r$.

   Consider the largest index i such that $d(a_i) \leq 2^n$.

   Now, note that, $d(a_i) > 2^{(n-1)^2}$ since otherwise we shall have,

$$d(a_i + 1) = rd(a_i) < 2^n \cdot 2^{(n-1)^2} = 2^{n^2-n+1} < 2^{n^2}$$

   , contradicting the maximality of i.

   A latter inequality is strict owing to $n > 1$ ($2^n > r > 1$).

   Observe that,

$$2^{n^2} < d(a_{i+1}) = rd(a_i) < 2^n \cdot 2^{n^2} < 2^{(n+1)^2}$$

   Now, since $2^{(n-1)^2} < d(a_i) \leq 2^{n^2}$ and $2^{n^2} < d(a_{i+1}) < 2^{(n+1)^2}$, a_i and a_{i+1} are of different colours.