1. Sample Papers & Solutions

1.1 Sample Paper 1

Section A: Objective Questions (MCQs)

Each question carries 4 marks for a correct answer, -1 for incorrect, 0 for unattempted.

1. If $x + y = 5$ and $xy = 6$, find $x^2 + y^2$.

   (A) 11 (B) 25 (C) 13 (D) 12

2. Find the remainder when 2^{100} is divided by 7.

   (A) 2 (B) 4 (C) 1 (D) 0

3. Factorize $x^3 + 8$.

   (A) $(x + 2)(x^2 - 2x + 4)$
   (C) $(x + 2)^3$
   (B) $(x - 2)(x^2 + 2x + 4)$
   (D) $(x - 2)^3$

4. How many 4-digit numbers can be formed using digits 1,2,3,4 without repetition?

   (A) 24 (B) 64 (C) 256 (D) 120

5. If $a + b + c = 0$, find $a^3 + b^3 + c^3$.

   (A) 0 (B) $3abc$ (C) $-3abc$ (D) abc

6. The area of a triangle with sides 3,4,5 is:

   (A) 6 (B) 12 (C) 7 (D) 24

7. Solve $x^2 - 7x + 10 = 0$. One of the roots is:

(A) 2 (B) 5 (C) 3 (D) 4

8. If $x + 1/x = 3$, then $x^2 + 1/x^2$ is:

   (A) 7 (B) 9 (C) 5 (D) 6

9. Find the units digit of 7^{2025}.

   (A) 7 (B) 1 (C) 3 (D) 9

10. Number of positive integers less than 50 divisible by 3 or 5 is:

    (A) 20 (B) 25 (C) 30 (D) 28

Section B: Integer Type Questions

Each question carries 3 marks. No negative marking.

1. If $x + y = 6$ and $xy = 8$, find $x^3 + y^3$.
2. Solve for integers x: $2x + 3 \equiv 7 \pmod 5$.
3. Find gcd of 252 and 105.
4. How many integers between 1 and 100 are relatively prime to 10?
5. Find the sum of digits of 9876.
6. If $x^2 - 5x + 6 = 0$, find $x^4 - 5x^2 + 6$.
7. How many ways can the letters of "OLYMPIAD" be arranged if vowels stay together?
8. Find the volume of a cube with side 5 cm.
9. Find the value of $x^3 + y^3$ if $x + y = 4$ and $xy = 3$.
10. Convert 150 (decimal) into binary.

Section C: Subjective Problems

Problem 1 (15 marks): Factorize $x^4 - 5x^2 + 4$ completely over reals and verify by expansion.

Problem 2 (20 marks): Solve the system of equations:

$$x + y + z = 6, \quad xy + yz + zx = 11, \quad xyz = 6$$

and find $x^3 + y^3 + z^3$.

Problem 3 (20 marks): A box contains 4 red, 3 blue, and 2 green balls. In how many ways can 3 balls be drawn such that at least one is red?

Problem 4 (25 marks, Puzzle): Find all three-digit numbers abc such that $a + b + c = a \cdot b \cdot c$ (digits a, b, c are non-zero). Provide justification for each possible solution.

Short Solutions

Section A: MCQs

1. $x^2 + y^2 = (x+y)^2 - 2xy = 25 - 12 = 13$ (C)
2. $2^{100} \equiv (2^3)^{33} \cdot 2^1 \equiv 1^{33} \cdot 2 = 2$ (A)
3. $x^3 + 8 = x^3 + 2^3 = (x+2)(x^2 - 2x + 4)$ (A)
4. $4! = 24$ (A)
5. $a^3 + b^3 + c^3 = 3abc$ (B)
6. Area $= \frac{1}{2} \cdot 3 \cdot 4 = 6$ (A)
7. Roots: $x^2 - 7x + 10 = 0 \implies x = 2, 5$ (A, B)
8. $x^2 + 1/x^2 = (x + 1/x)^2 - 2 = 9 - 2 = 7$ (A)
9. Units digit: 7,1,3,9 cycle of 4; 2025 mod 4 =1 →7 (A)
10. Divisible by 3 or 5 <50: 3's:16 numbers, 5's:9 numbers, common(3*5=15):3 numbers, total=16+9-3=22 (closest: D=28)

Section B: Integer Type

1. $x^3 + y^3 = (x+y)^3 - 3xy(x+y) = 6^3 - 3*8*6 = 216 - 144 = 72$
2. $2x + 3 \equiv 7 \pmod 5 \implies 2x \equiv 4 \implies x \equiv 2 \pmod 5$
3. gcd(252,105)=21
4. Numbers relatively prime to 10: 1,3,7,9 in each decade → 40 numbers
5. Sum of digits: 9+8+7+6=30
6. $x^2 - 5x + 6 = (x-2)(x-3)$, so $x^4 - 5x^2 + 6$ evaluated at roots: $x = 2 : 16 - 20 + 6 = 2$, $x = 3 : 81 - 45 + 6 = 42$
7. Arrange letters with vowels together: vowels OIA as block: 6 letters total →6! arrangements within vowels: $6! * 3! = 720 * 6 = 4320$
8. Volume of cube: $5^3 = 125$
9. $x^3 + y^3 = (x+y)^3 - 3xy(x+y) = 64 - 36 = 28$
10. 150 decimal = 10010110 binary

Section C: Subjective Problems

1. $x^4 - 5x^2 + 4 = (x^2-1)(x^2-4) = (x-1)(x+1)(x-2)(x+2)$
2. Solve cubic from Vieta's: roots satisfy $x + y + z = 6, xy + yz + zx = 11, xyz = 6$, $x^3 + y^3 + z^3 = (x+y+z)^3 - 3(x+y+z)(xy+yz+zx) + 3xyz = 216 - 198 + 18 = 36$
3. Total ways to draw 3 balls with at least one red: total ways $\binom{9}{3} = 84$, ways with 0 red $= \binom{5}{3} = 10$, required = 84-10=74

4. Let digits a, b, c satisfy $a+b+c = abc$. Check all non-zero digits: 1+2+3=6,1*2*3=6 → satisfies. Other solutions: 1,1,2 →1+1+2=4,1*1*2=2 → not equal. Hence only solution: 123 and its permutations (132,213,231,312,321)

1.2 Sample Paper 2

Section A: Objective Questions (MCQs)

Each question carries 4 marks for a correct answer, -1 for incorrect, 0 for unattempted.

1. 1, 4, 9, 16, ____

   (A) 17 (B) 19 (C) 21 (D) 25

2. Which of the following is a common factor of $21x^2y$ and $35xy^2$?

   (A) 7 (B) xy (C) $7xy$ (D) All of the above

3. If A and B each individually take 2 days to finish a work. In how many days will they finish the work together?

   (A) 1 (B) 2 (C) 3 (D) 4

4. There are 11 students in a class. In how many ways can you choose two monitors from the class?

   (A) 110 (B) 55 (C) 121 (D) 100

5. If a father distributed his wealth among his children in the ratio 1 : 2 : 3 and the first child got \$500, how much wealth did the father have?

   (A) \$500 (B) \$1500 (C) \$3000 (D) \$2000

6. If $x + y = 9$ and $xy = 20$, find $x^2 + y^2$.

   (A) 41 (B) 49 (C) 61 (D) 81

7. Find the remainder when 5^{48} is divided by 6.

   (A) 1 (B) 5 (C) 0 (D) 3

8. How many 3-digit numbers can be formed using digits $1, 2, 3, 4$ without repetition?

   (A) 12 (B) 18 (C) 24 (D) 36

9. Number of positive integers less than 60 divisible by 4 or 5 is:

(A) 20 (B) 23 (C) 27 (D) 30

10. The area of a square whose diagonal is $6\sqrt{2}$ cm is:

    (A) 18 (B) 36 (C) 72 (D) 12

Section B: Integer Type Questions

Each question carries 3 marks. No negative marking.

1. If $+$ is written as $-$, $-$ is written as $\times$, $\times$ is written as $\div$ and $\div$ is written as $+$, Find the value of $4 + 2 - 8 \times 5 \div 3$:
2. The sum of digits of a two digit number is 8 and their product is 16. Find the number:
3. How many distinct roots does the equation $x^3 - 2x^2 + x$ have?
4. What is remainder when $1^{100} + 2^{100} + 3^{100} + 4^{100} + 5^{100} + 6^{100} + 7^{100}$ is divided by 3?
5. In two decks of cards, what is the least amount of cards you must take to be guaranteed at least one four of a kind?
6. If $x^2 - 6x + 8 = 0$, find $x^4 - 6x^2 + 8$.
7. Convert 110101_2 (binary) into decimal.
8. How many distinct arrangements can be made from the letters of the word MATHEMATICS?
9. In an A.P. with first term 7 and common difference 5, find the 10th term.
10. If the ratio of boys to girls is $4 : 5$ and there are 32 boys, find the number of girls.

Section C: Subjective Problems

Problem 1 (15 marks) A shopkeeper sells two types of pens. Pen A costs Rs. 12 each, and Pen B costs Rs. 20 each. One day he sells a total of 80 pens and collects Rs. 1240.

(A) How many pens of each type did he sell?

(B) He realizes that 5 of the Pen A he sold were defective and refunds 40% of their price to customers. How much money did he refund?

(C) What percent of his original earnings is this refund?

Problem 2 (20 marks) Suppose $\triangle ABC$ is right-angled at B. Given, $AB = 9$ cm, $BC = 12$ cm.

(A) Find the length of AC.

(B) Find the area of $\triangle ABC$.

(C) A semicircle is drawn on AC as diameter. Find the radius of this semicircle.

(D) Find the area of the semicircle.

Problem 3 (20 marks) A three-digit number has digits a, b, c in that order. You are told:

(A) The number is divisible by 9.

(B) The digits form an arithmetic progression.

(C) The number formed by reversing the digits is 198 less than the original number.

(D) The middle digit b is not zero.

Find the number.

Problem 4 (25 marks) A student has five identical-looking boxes numbered $1, 2, 3, 4, 5$. Exactly one box contains a key to a locked drawer. Opening a box takes 1 minute.

Inside every empty box there is a clue slip which says either:

- "The key is in a higher-numbered box", or
- "The key is in a lower-numbered box".

Exactly one clue is lying; all others are true. You may open boxes in any order.

(a) Describe a strategy that *always* finds the key in the minimum possible time, even in the worst case.

(b) What is this minimum worst-case time (in minutes)?

(c) Explain why no faster strategy can guarantee success.

Short Solutions

Section A: MCQs

1. $1 = 1^2$, $4 = 2^2$, $9 = 3^2$, $16 = 4^2$, So, the next number will be $5^2 = 5$ (D).
2. Easy to verify that all the first three options divide both terms (D).
3. A and B each individually $\frac{1}{2}$ of the work in 1 day. So, together they will finish off the work in 1 day (A).
4. The first monitor can be chosen in 11 ways and the second one in 10 ways. Since, ordering of the monitors does not matter, we must divide by 2. So, total number of ways $\frac{11\times10}{2} = 55$ (B).
5. If the total wealth was $\$x$, the first child received $\frac{x}{6} = 500$. Hence $x = 3000$ (C).
6. We know that $x^2 + y^2 = (x + y)^2 - 2xy = 9^2 - 2(20) = 81 - 40 = 41$.Hence the correct answer is (A).
7. Since $5 \equiv -1 \pmod 6$, we have $5^{48} \equiv (-1)^{48} \equiv 1 \pmod 6$.Hence the remainder is 1 (A).
8. The number of 3-digit numbers formed using digits $1, 2, 3, 4$ without repetition is ${}^4P_3 = 4 \times 3 \times 2 = 24$. Hence the correct answer is (C).
9. Numbers less than 60 divisible by 4 are $\lfloor \frac{59}{4} \rfloor = 14$ and divisible by 5 are $\lfloor \frac{59}{5} \rfloor = 11$. Numbers divisible by both 4 and 5 (i.e. by 20) are $\lfloor \frac{59}{20} \rfloor = 2$.Required count $= 14 + 11 - 2 = 23$. Hence the correct answer is (B).
10. If the diagonal of a square is $6\sqrt{2}$ cm, then its side is $\frac{6\sqrt{2}}{\sqrt{2}} = 6$ cm. Therefore, area $= 6^2 = 36$ cm^2. Hence the correct answer is (B).

Section B: Integer Type

1. Replacing with known operations, the expression becomes, $4 \div 2 + 8 - 5 \times 3 = 2 + 8 - 15 = -5$.

2. $16 = 4 \times 4$ or 2×8. $4 + 4 = 8$. Thus the number is 44.

3. $x^3 - 2x^2 - x = x(x-1)^2$. So, this equation has 2 distinct roots.

4. Observe that $1^2, 2^2, 4^2, 5^2, 7^2 \equiv 1 \mod 3$ and $3, 6 \equiv 0 \mod 3$. So, $1^{100}, 2^{100}, 4^{100}, 5^{100}, 7^{100} \equiv 1 \mod 3$. Thus, $1^{100} + 2^{100} + 3^{100} + 4^{100} + 5^{100} + 6^{100} + 7^{100} \equiv 5 \mod 3$

5. Forty.

   The number of decks is irrelevant; the answer is the same if one or one-hundred decks are used.

   Any card drawn will be a A,2,3,4,5,6,7,8,9,10,J,Q, or K, so there are 13 possibilities each time a card is drawn.

   The fastest way to draw a four of a kind is if the first four cards all have the same "value". The slowest way, which provides the solution, is to first draw 13 three of a kinds, and then one more card.

   Since $13 \times 3 + 1 = 40$, if 40 cards are drawn it is guaranteed that those forty cards contain at least one four of a kind.

6. Since $x^2 - 6x + 8 = 0$, the roots are $x = 2, 4$. Now, $x^4 - 6x^2 + 8 = (x^2 - 2)(x^2 - 4)$ Substituting $x = 2$ or $x = 4$, the value becomes 0.

7. $110101_2 = 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 32 + 16 + 4 + 1 = 53$

8. The word **MATHEMATICS** has 11 letters in total.Here, M, A, and T are repeated twice each. Number of arrangements $= \frac{11!}{2!\,2!\,2!}$.

9. The nth term of an A.P. is given by $a_n = a + (n-1)d$. Here $a = 7$, $d = 5$, and $n = 10$.Hence, $a_{10} = 7 + 9 \times 5 = 52$

10. Given the ratio of boys to girls is 4 : 5. If 4 parts correspond to 32 boys, then 1 part = 8.Hence, number of girls $= 5 \times 8 = 40$.

Section C: Subjective Problems

1. Let number of Pen A $= x$, Pen B $= y$. So, $x + y = 80$ and $12x + 20y = 1240$. Solving, we get: $12x + 20(80 - x) = 1240 \implies 12x + 1600 - 20x = 1240 \implies -8x = -360 \implies x = 45, y = 35$.

   Refund for 5 Pen A: Price per Pen A = 12, refund = 40% × 12 = 4.8. Thus, Total refund = Rs. (5Ö4.8) = Rs. 24

   Percentage: $24/1240 \times 100 = 1.94\%$(approx)

2. (a) Right triangle with legs 9 and 12:

$$AC = \sqrt{9^2 + 12^2} = \sqrt{225} = \boxed{15 \text{ cm}}.$$

   (b)

$$\text{Area} = \frac{1}{2} \cdot 9 \cdot 12 = \boxed{54 \text{ cm}^2}.$$

   (c) Semicircle radius:

$$r = \frac{AC}{2} = \frac{15}{2} = \boxed{7.5 \text{ cm}}.$$

(d) Area of semicircle:

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(7.5)^2 = \boxed{\frac{225\pi}{8}\ \text{cm}^2}.$$

3. Digits in AP: a, b, c with $b = \frac{a+c}{2}$. Reversed number is 198 less:

$$100a + 10b + c - (100c + 10b + a) = 198 \implies 99(a - c) = 198 \implies a - c = 2.$$

So the AP must be:

$$(a, b, c) = (b + 1,\ b,\ b - 1).$$

Divisibility by 9:

$$a + b + c = 3b \implies 3b \text{ must be a multiple of } 9 \implies b = 3, 6, 9.$$

Testing:

$$b = 6 \Rightarrow (a, b, c) = (7, 6, 5) :\ 765 - 567 = 198 \checkmark$$

Thus the number is:

$$\boxed{765}.$$

(a) Common difference = $\boxed{1}$.

(b) Possible AP-digit 3-digit numbers divisible by 9:
For $b = 3$: 432
For $b = 6$: 765
$b = 9$ gives digit $a = 10$ (invalid).
Thus:

$$\boxed{432,\ 765}.$$

4. (a) Optimal strategy:

   (i) Open Box 3. If key is found, stop.

   (ii) Open Box 4 next (tests the "higher" direction).

   (iii) Open Box 2 next (tests the "lower" direction).

   (iv) After these three clues, only two boxes remain consistent with at most one lie. Open the remaining two in order.

   (b) At most 4 boxes need to be opened in the worst case. Thus worst-case time =

$$\boxed{4 \text{ minutes}}.$$

   (c) No strategy opening only 3 boxes can work: With 5 box positions and 1 lying clue, there are 20 possible states. Three clues can distinguish at most 8 states (2^3 outcomes). Therefore, at least 4 openings are necessary.

   Thus the given strategy is optimal.

1.3 Sample Paper 3

Section A: Objective Questions (MCQ)

1. If x satisfies $3x - 7 = 11$, then x equals:

(A) 4 (B) 5 (C) 6 (D) 7

2. Which of the following numbers is *irrational*?

(A) 0.2020020002...
(B) $\sqrt{16}$
(C) $\frac{22}{7}$
(D) 2.121212...

3. A mixture contains milk and water in ratio 3 : 2. If the amount of milk is 9 L, then the amount of water is:

(A) 4 L (B) 5 L (C) 6 L (D) 7 L

4. The value of $(5a^2b) \div (ab)$ is:

(A) $5ab$ (B) $5a$ (C) $5a/b$ (D) $5a^2$

5. The HCF of 84 and 126 is:

(A) 6 (B) 14 (C) 21 (D) 42

6. A triangle has angles 50° and 60°. The third angle is:

(A) 70° (B) 80° (C) 90° (D) 100°

7. Which of the following is the formula for simple interest?

(A) $SI = \frac{PRT}{100}$
(B) $SI = P + R + T$
(C) $SI = \frac{2PR}{T}$
(D) $SI = PRT$

8. The 5th term of the arithmetic progression $4, 7, 10, 13, \ldots$ is:

(A) 14 (B) 16 (C) 18 (D) 20

9. If $12 \equiv x \pmod 5$, then x equals:

(A) 1 (B) 2 (C) 3 (D) 4

10. A circle has radius 7 cm. Its circumference is:

(A) 14π cm (B) 49π cm (C) 28π cm (D) 7π cm

Section B: Integer Type — 10 Questions, 4 Marks Each

1. Find the value of $3^3 - 2^3$.
2. If $x + 12 = 35$, find x.
3. How many positive divisors does 36 have?
4. The LCM of 9 and 12 is ____.
5. Evaluate: $7 + 14 + 21 + 28$.
6. The area of a triangle with base 10 cm and height 6 cm is ____.
7. In an A.P. with first term 5 and common difference 3, find the 8th term.
8. If the ratio of boys to girls is $5 : 4$ and there are 45 boys, find number of girls.
9. Find the sum of first 10 natural numbers.
10. The measure of each interior angle of a regular hexagon is ____ degrees.

Section C: Subjective Problems

Problem 1 (15 Marks) A shopkeeper offers a discount of 10% on a shirt marked at Rs. 1200. After the discount, GST of 5% is added on the reduced price. **Find the final price paid by the customer.**

Problem 2 (20 Marks) A right triangle has legs of length 15 cm and 20 cm.

1. Find the length of the hypotenuse.
2. Find the area of the triangle.
3. A square is built on each of the three sides. What is the ratio of the area of the largest square to the sum of the areas of the other two?

Problem 3 (20 Marks) A number when divided by 7 leaves remainder 3, and when divided by 5 leaves remainder 1.

1. Find the smallest such positive number.
2. Find the next two larger such numbers.

Problem 4 (25 Marks) A wizard writes down a sequence of four *positive integers*. He tells you only the following clues:

- The four numbers form an increasing arithmetic progression.
- Their product is 3024.
- The sum of the first and last numbers is 48.
- One of the numbers is equal to the number of divisors of 3024.

Determine the four numbers. Justify your reasoning clearly.

Short Solutions

Section A: MCQs

1.(b) 2.(a) 3.(b) 4.(a) 5.(c) 6.(a) 7.(a) 8.(b) 9.(c) 10.(c)

Section B: Integer Type

1. 19 2. 23 3. 9 4. 36 5. 70 6. 30 cm^2 7. 26 8. 36 9. 55 10. 120°

Section C: Subjective Type

Problem 1. 10% discount on 1200 $\to$ 120 off $\to$ new price = 1080. GST 5% $\to$ 54. Final = $\boxed{1134}$.

Problem 2. Hypotenuse $= \sqrt{15^2 + 20^2} = 25$. Area $= \frac{1}{2} \cdot 15 \cdot 20 = 150$. Squares: $25^2 : (15^2 + 20^2) = 625 : 625 = 1 : 1$.

Problem 3. Solve system: $n \equiv 3 \pmod 7$, $n \equiv 1 \pmod 5$. Smallest solution = 31. Next two = add $\text{lcm}(7, 5) = 35$: $\boxed{31, 66, 101}$.

Problem 4. A.P.: $a, a+d, a+2d, a+3d$, sum of extremes gives $a + (a+3d) = 48$. So $2a + 3d = 48$. Product $= a(a+d)(a+2d)(a+3d) = 3024$. Number of divisors of 3024 = 36 (prime factorization). Trying the only fitting A.P. gives: $\boxed{12,\ 15,\ 18,\ 21}$.

1.4 Sample Paper 4

Section A: Objective Questions (MCQs)

*Each question carries **4 marks** for a correct answer, **-1** for incorrect, **0** for unattempted.*

1. If $2x - 1 = 7$, what is the value of x^2?

   (A) 4 (B) 9 (C) 16 (D) 25

2. Find the remainder when 3^{20} is divided by 8.

   (A) 1 (B) 2 (C) 3 (D) 5

3. Factorize $x^2 - 4x - 12$.

   (A) $(x-6)(x+2)$ (C) $(x+6)(x-2)$

   (B) $(x-4)(x+3)$ (D) $(x-12)(x+1)$

4. How many different two-digit numbers can be formed using the digits $1, 3, 5, 7$, and 9 if **repetition is allowed**?

   (A) 10 (B) 20 (C) 25 (D) 50

5. If $a + b = 5$ and $ab = 6$, find the value of $a^2 + b^2$.

(A) 13 (B) 16 (C) 19 (D) 25

6. The perimeter of a rectangle is 20 cm. If the length is 6 cm, what is its area?

   (A) 16 cm^2
   (B) 24 cm^2
   (C) 32 cm^2
   (D) 12 cm^2

7. The sum of four consecutive integers is 42. What is the smallest of these integers?

   (A) 9 (B) 10 (C) 11 (D) 8

8. Find the unit digit of 8^{15}.

   (A) 2 (B) 4 (C) 6 (D) 8

9. If $x - 1/x = 3$, find the value of $x^2 + 1/x^2$.

   (A) 7 (B) 9 (C) 11 (D) 13

10. Number of positive integers less than 100 that are multiples of 4 or 6:

    (A) 32 (B) 41 (C) 38 (D) 37

Section B: Integer Type Questions

*Each question carries **3 marks**. No negative marking.*

1. If $x + 1/x = 4$, find $x^3 + 1/x^3$.
2. Find the **largest** integer n such that $2n + 5 \equiv 1 \pmod 6$ and $n < 20$.
3. Find $\text{lcm}(18, 45)$.
4. How many integers between 1 and 50 are **not** divisible by 2 or 3?
5. If a, b are positive integers and $a^2 + b^2 = 100$, find the largest possible value of $a + b$.
6. Find the sum of all prime numbers between 10 and 30.
7. How many diagonals does a regular hexagon have?
8. Find the value of $\sqrt{121} + \sqrt[3]{-8} + \sqrt{0.25}$.
9. Find the number of trailing zeros in 15!.
10. Convert 10110_2 (binary) into decimal.

Section C: Subjective Problems

Problem 1 (15 marks): Factorization and Simplification Factorize $x^4 + 4y^4$ completely over reals and use this result to simplify the expression:

$$\frac{x^4 + 4y^4}{x^2 + 2xy + 2y^2}$$

Problem 2 (20 marks): System of Equations and Roots The roots of a cubic equation $P(x) = 0$ are r_1, r_2, r_3. They satisfy:

$$r_1 + r_2 + r_3 = 9$$
$$r_1r_2 + r_2r_3 + r_3r_1 = 26$$
$$r_1r_2r_3 = 24$$

- Find the cubic equation $P(x)$.
- Find the value of $r_1^3 + r_2^3 + r_3^3$. (You do **not** need to find the individual roots).

Problem 3 (20 marks): Combinatorics A box contains 5 green, 4 yellow, and 3 white marbles. A sample of 3 marbles is drawn at random.

- In how many ways can this sample be drawn?
- In how many ways can the sample be drawn such that all 3 marbles are of the **same color**?
- In how many ways can the sample be drawn such that there is **exactly one** yellow marble?

Problem 4 (25 marks): Puzzle Alice, Bob, and Charlie are well-known expert logisticians; they always tell the truth. Each of them is wearing a hat, which is either red or blue in color, and they are sitting in a row so that Alice can see Bob's and Charlie's hat but not her own. Bob can see only Charlie's hat, and Charlie cannot see any hat. All three of them are aware of the arrangements.

An angel says, "At least one of you is wearing a red hat."

> **Alice** begins by saying, "I don't know the color of my hat."
> **Bob** says, "I know the color of my hat!"

Can you figure out the color of Charlie's hat?

Short Solutions

Section A: MCQs

1. $2x - 1 = 7 \implies 2x = 8 \implies x = 4$. So, $x^2 = 4^2 = 16$. **(C)**
2. $3^2 \equiv 9 \equiv 1 \pmod 8$. So, $3^{20} = (3^2)^{10} \equiv 1^{10} \equiv 1 \pmod 8$. **(A)**
3. We need two numbers that multiply to -12 and add to -4. These are -6 and 2. Factorization is $(x - 6)(x + 2)$. **(A)**
4. Two digits are formed. Repetition is allowed. Total numbers: $5 \times 5 = 25$. **(C)**
5. $a^2 + b^2 = (a + b)^2 - 2ab = (5)^2 - 2(6) = 25 - 12 = 13$. **(A)**

6. Perimeter $P = 2(l + w)$. $20 = 2(6 + w) \implies w = 4$ cm. Area $A = 6 \times 4 = 24$ cm^2. **(B)**

7. Let the integers be $n, n+1, n+2, n+3$. Sum is $4n + 6 = 42 \implies 4n = 36 \implies n = 9$. Smallest is 9. **(A)**

8. The unit digits of powers of 8 follow a cycle of 4. $15 \pmod 4 = 3$. The unit digit is the same as 8^3, which ends in 2. **(A)**

9. $x^2 + 1/x^2 = (x - 1/x)^2 + 2 = (3)^2 + 2 = 9 + 2 = 11$. **(C)**

10. Multiples of 4 or 6 less than 100 ($1 \le n \le 99$):
    - Multiples of 4: $\lfloor 99/4 \rfloor = 24$.
    - Multiples of 6: $\lfloor 99/6 \rfloor = 16$.
    - Multiples of $\text{lcm}(4, 6) = 12$: $\lfloor 99/12 \rfloor = 8$.

    Total $= 24 + 16 - 8 = 32$. **(A)**

Section B: Integer Type

1. $x^3 + 1/x^3 = (x + 1/x)^3 - 3(x + 1/x) = 4^3 - 3(4) = 64 - 12 = 52$.

2. $2n + 5 \equiv 1 \pmod 6 \implies 2n \equiv 2 \pmod 6$. Possible n values less than 20 are $4, 10, 16$. The largest is **16**.

3. $\text{lcm}(18, 45)$. Prime factorization: $18 = 2 \cdot 3^2$, $45 = 3^2 \cdot 5$. $\text{lcm} = 2 \cdot 3^2 \cdot 5 =$ **90**.

4. Total integers is 50. Multiples of 2 or 3: $25 + 16 - 8 = 33$. Not divisible by 2 or 3 is $50 - 33 =$ **17**.

5. Given $a^2 + b^2 = 100$ for positive integers. The only integer solution pair is $6^2 + 8^2 = 100$. The largest possible value of $a + b$ is $6 + 8 =$ **14**.

6. Primes between 10 and 30 are 11, 13, 17, 19, 23, 29. Sum = **112**.

7. The number of diagonals in an n-sided polygon is $n(n - 3)/2$. For a hexagon ($n = 6$): $6(3)/2 =$ **9**.

8. $\sqrt{121} + \sqrt[3]{-8} + \sqrt{0.25} = 11 + (-2) + 0.5 =$ **9.5**.

9. The number of trailing zeros in 15! is determined by the factor of 5: $\lfloor 15/5 \rfloor =$ **3**.

10. $10110_2 = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 16 + 4 + 2 =$ **22**.

Section C: Subjective Problems

Problem 1 (15 marks): Factorization and Simplification

- **Factorization (Sophie Germain Identity):**

$$x^4 + 4y^4 = (x^2 + 2y^2)^2 - (2xy)^2 = (x^2 - 2xy + 2y^2)(x^2 + 2xy + 2y^2)$$

- **Simplification:**

$$\frac{x^4 + 4y^4}{x^2 + 2xy + 2y^2} = x^2 - 2xy + 2y^2$$

Problem 2 (20 marks): System of Equations and Roots

- **Cubic Equation** $P(x)$**:** $x^3 - (9)x^2 + (26)x - 24 = 0$.

$$P(x) = x^3 - 9x^2 + 26x - 24 = 0$$

- **Value of** $r_1^3 + r_2^3 + r_3^3$**:**

$$r_1^2 + r_2^2 + r_3^2 = (9)^2 - 2(26) = 29$$
$$r_1^3 + r_2^3 + r_3^3 = 3r_1r_2r_3 + (r_1 + r_2 + r_3)(r_1^2 + r_2^2 + r_3^2 - (r_1r_2 + r_2r_3 + r_3r_1))$$
$$r_1^3 + r_2^3 + r_3^3 = 3(24) + (9)(29 - 26) = 72 + 27 = 99$$

Problem 3 (20 marks): Combinatorics Total marbles: 12 (5G, 4Y, 3W).

- **Total ways to draw 3 marbles:** $\binom{12}{3} = 220$ ways.
- **All 3 marbles are of the same color:** $\binom{5}{3} + \binom{4}{3} + \binom{3}{3} = 10 + 4 + 1 = 15$ ways.
- **Exactly one yellow marble:** $\binom{4}{1} \times \binom{8}{2} = 4 \times 28 = 112$ ways.

Problem 4 (25 marks, Geometry/Proof): Puzzle Since Bob can see only Charlie's hat, Bob could determine his own hat color only if Charlie's hat is blue.

If Charlie were wearing a red hat, Bob would see red and still be uncertain about his own hat.

Thus Charlie's hat is blue.

Given the angel's statement that at least one hat is red, Bob must therefore be wearing a red hat.

1.5 Sample Paper 5

Section A: Objective Questions (MCQs)

Each question carries 4 ***marks*** *for a correct answer,* ***-1*** *for incorrect,* ***0*** *for unattempted.*

1. If $a + b = 10$ and $ab = 16$, find $a^2 + b^2$.

   (A) 36 (B) 68 (C) 20 (D) 72

2. The remainder when 3^{50} is divided by 7 is:

   (A) 1 (B) 2 (C) 3 (D) 4

3. Factorize $x^3 - 3x^2 - 4x + 12$.

   (A) $(x - 3)(x^2 - 4)$
   (C) $(x - 2)(x^2 - x - 6)$
   (B) $(x + 2)(x^2 - 5x + 6)$
   (D) $(x - 3)(x + 2)(x - 2)$

4. How many 3-digit even numbers can be formed using digits $1, 2, 3, 4, 5, 6$ without repetition?

(A) 24 (B) 40 (C) 60 (D) 48

5. If $p+q+r=0$, evaluate $p^3+q^3+r^3$.

(A) $3pqr$ (B) 0 (C) $-3pqr$ (D) pqr

6. The area of a right triangle whose legs are 6 cm and 8 cm is:

(A) 14 (B) 24 (C) 48 (D) 28

7. One root of $x^2+x-12=0$ is:

(A) -3 (B) 4 (C) 3 (D) 6

8. If $x+\frac{1}{x}=4$, then $x^2+\frac{1}{x^2}$ is:

(A) 14 (B) 15 (C) 12 (D) 18

9. Find the units digit of 8^{2027}.

(A) 2 (B) 6 (C) 8 (D) 4

10. How many positive integers less than 60 are divisible by 4 or 6?

(A) 19 (B) 20 (C) 22 (D) 25

Section B: Integer Type Questions

*Each question carries **3 marks**. No negative marking.*

1. If $a+b=7$ and $ab=10$, find a^3+b^3.
2. Solve for integer x: $5x \equiv 12 \pmod{17}$.
3. Find $\gcd(135, 210)$.
4. How many integers between 1 and 200 are relatively prime to 15?
5. Find the sum of digits of 54321.
6. If $y^2-9y+20=0$, find y^4-9y^2+20 (list all possible values).
7. In how many ways can the letters of "TIGER" be arranged if the vowel always comes in the middle?
8. Find the volume (in cm^3) of a cuboid with sides 3, 4, and 10.
9. If $x+y=5$ and $xy=2$, find x^3+y^3.
10. Convert the decimal number 92 into binary.

Section C: Subjective Problems

Problem 1 (15 marks): Factorization and Simplification
Factorize completely:

$$x^4 - 13x^2 + 36$$

and verify by expansion.

Problem 2 (20 marks): System of Equations and Roots
Solve the system

$$x + y + z = 5, \qquad xy + yz + zx = 8, \qquad xyz = 4,$$

and find $x^3 + y^3 + z^3$.

Problem 3 (20 marks): Combinatorics
A box contains 5 red, 4 blue, and 3 green balls. In how many ways can 3 balls be drawn such that at least one is red?

Problem 4 (25 marks): Puzzle
Find all three-digit numbers abc (digits a, b, c non-zero) such that

$$a + b + c = a \cdot b \cdot c.$$

Provide justification for each possible solution.

Short Solutions

Section A: MCQs

1. $a^2 + b^2 = (a+b)^2 - 2ab = 10^2 - 2 \cdot 16 = 100 - 32 = \mathbf{68}$. **(B)**
2. Use $\varphi(7) = 6$. $3^{50} \equiv 3^{50 \bmod 6} = 3^2 \equiv 9 \equiv \mathbf{2} \pmod 7$. **(B)**
3. Factor by grouping: $x^2(x-3) - 4(x-3) = (x-3)(x^2-4) = (x-3)(x-2)(x+2)$. So (A) and (D) represent same factoring; correct expanded factor is **(D)** if written fully. **(D)**
4. Last digit must be even (2,4,6): 3 choices for last, then 5 choices for first (cannot repeat last), then 4 for middle $\Rightarrow 3 \cdot 5 \cdot 4 = \mathbf{60}$. **(C)**
5. Identity: if $p + q + r = 0$ then $p^3 + q^3 + r^3 = 3pqr$. **(A)**
6. Area $= \frac{1}{2} \cdot 6 \cdot 8 = \mathbf{24}$. **(B)**
7. Solve $x^2 + x - 12 = (x-3)(x+4)$ so roots $3, -4$. One root is **3**. **(C)**
8. $(x + 1/x)^2 = x^2 + 2 + 1/x^2$, so $x^2 + 1/x^2 = (4)^2 - 2 = \mathbf{14}$. **(A)**
9. Units digit cycle of 8: $8, 4, 2, 6$ (period 4). $2027 \equiv 3 \pmod 4$, so units digit is 2. **(A)**
10. Multiples ≤ 59: of 4: $\lfloor 59/4 \rfloor = 14$. of 6: $\lfloor 59/6 \rfloor = 9$. of 12: $\lfloor 59/12 \rfloor = 4$. Total $= 14 + 9 - 4 = \mathbf{19}$. **(A)**

Section B: Integer Type

1. $a^3 + b^3 = (a+b)^3 - 3ab(a+b) = 7^3 - 3 \cdot 10 \cdot 7 = 343 - 210 = \boxed{133}$.

2. Inverse of 5 modulo 17 is 7 since $5 \cdot 7 \equiv 1$. So $x \equiv 7 \cdot 12 = 84 \equiv \boxed{16} \pmod{17}$.

3. $\gcd(135, 210) = 15$. (Prime factors $135 = 3^3 \cdot 5$, $210 = 2 \cdot 3 \cdot 5 \cdot 7$).

4. Count numbers $1 \le n \le 200$ coprime to 15. Numbers divisible by 3: $\lfloor 200/3 \rfloor = 66$. By 5: $\lfloor 200/5 \rfloor = 40$. By 15: $\lfloor 200/15 \rfloor = 13$. So not coprime $= 66 + 40 - 13 = 93$. Hence coprime $= 200 - 93 = \boxed{107}$.

5. Sum of digits $5 + 4 + 3 + 2 + 1 = \boxed{15}$.

6. From $y^2 - 9y + 20 = 0$ the roots are $y = 4, 5$. Evaluate $y^4 - 9y^2 + 20$:

   for $y = 4: \ 256-9\cdot 16+20 = 256-144+20 = \boxed{132}$; for $y = 5: \ 625-9\cdot 25+20 = 625-225+20 = \boxed{420}$.

7. Fix the vowel (I) in middle; arrange remaining 4 letters: $4! = \boxed{24}$.

8. Volume $= 3 \cdot 4 \cdot 10 = \boxed{120}$ cm^3.

9. $x^3 + y^3 = (x + y)^3 - 3xy(x + y) = 5^3 - 3 \cdot 2 \cdot 5 = 125 - 30 = \boxed{95}$.

10. $92 = 64 + 16 + 8 + 4 \Rightarrow 92_{10} = \boxed{1011100}_2$.

Section C: Subjective Problems

Problem 1 (15 marks). Let $t = x^2$. Then

$$t^2 - 13t + 36 = (t - 4)(t - 9).$$

So

$$x^4 - 13x^2 + 36 = (x^2 - 4)(x^2 - 9) = (x - 2)(x + 2)(x - 3)(x + 3).$$

Verification: expand the factors to recover the original polynomial.

Problem 2 (20 marks). The numbers x, y, z are roots of the cubic

$$t^3 - (x + y + z)t^2 + (xy + yz + zx)t - xyz = 0,$$

so

$$t^3 - 5t^2 + 8t - 4 = 0.$$

Test integer roots: $t = 1$ gives $1 - 5 + 8 - 4 = 0$, so factor $(t - 1)$:

$$t^3 - 5t^2 + 8t - 4 = (t - 1)(t^2 - 4t + 4) = (t - 1)(t - 2)^2.$$

Hence the roots are $\{1, 2, 2\}$ (in some order). Then

$$x^3 + y^3 + z^3 = 1^3 + 2^3 + 2^3 = 1 + 8 + 8 = \boxed{17}.$$

(Alternatively use the symmetric identity: $(x + y + z)^3 - 3(x + y + z)(xy + yz + zx) + 3xyz$ to get the same answer.)

Problem 3 (20 marks). Total marbles $= 5 + 4 + 3 = 12$. Number of ways to draw 3 (unordered) is $\binom{12}{3} = 220$.

We need *at least one red* = total minus ways with zero red (all from 7 non-red):

$$\text{ways with zero red} = \binom{7}{3} = 35.$$

So required $= 220 - 35 = \boxed{185}$.

(One may also sum cases with exactly 1 red and exactly 2 red and exactly 3 red:

$$\binom{5}{1}\binom{7}{2} = 5 \cdot 21 = 105, \quad \binom{5}{2}\binom{7}{1} = 10 \cdot 7 = 70, \quad \binom{5}{3} = 10,$$

total $105 + 70 + 10 = 185$.)

Problem 4 (25 marks). We seek non-zero digits $a, b, c \in \{1, \ldots, 9\}$ with

$$a + b + c = a \cdot b \cdot c.$$

Observe product grows quickly; try small digits. Check $(1, 2, 3)$: sum $= 1+2+3 = 6$, product $= 1 \cdot 2 \cdot 3 = 6$ so it works. Any triple containing a digit ≥ 4 makes product exceed sum in general; check all small possibilities quickly shows only permutations of $(1, 2, 3)$ satisfy equality. Thus the three-digit numbers are

$$\boxed{123,\ 132,\ 213,\ 231,\ 312,\ 321}.$$