SPARQ Senior 2026 Solutions

Time: 60 minutes **Date: January 04, 2026**

MCQ 1

You are in line at an ice cream shop when the manager announces that she will give a free sundae to the first person in line whose birthday is the same as someone who has already bought an ice cream. Assuming that you do not know anyone else's birthday and that all birthdays are uniformly distributed across the 365 days in a normal year, what position in line will you choose to maximize your probability of receiving the free sundae?

(A) 19

(B) 20

(C) 21

(D) None of these.

Solution. Let $P(n)$ be the probability that the n-th person in line is the first person to share a birthday with someone previously in line.

For the n-th person to be the first winner, two independent conditions must be met:

1. The first $n-1$ people must all have distinct birthdays.
2. The n-th person must have a birthday that matches one of the $n-1$ birthdays already seen.

The probability that the first $n-1$ people have unique birthdays is:

$$P(\text{unique}_{n-1}) = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{365-(n-2)}{365}$$

The probability that the n-th person matches one of those $n-1$ birthdays is:

$$P(\text{match}) = \frac{n-1}{365}$$

Thus, the probability of winning at position n is:

$$P(n) = \left(\prod_{i=0}^{n-2} \frac{365-i}{365} \right) \times \frac{n-1}{365}$$

To find the maximum, we examine the ratio $\frac{P(n+1)}{P(n)}$. The probability $P(n)$ is increasing as long as this ratio is greater than 1.

$$\frac{P(n+1)}{P(n)} = \frac{\left(\prod_{i=0}^{n-1} \frac{365-i}{365}\right) \times \frac{n}{365}}{\left(\prod_{i=0}^{n-2} \frac{365-i}{365}\right) \times \frac{n-1}{365}}$$

After cancelling common terms:

$$\frac{P(n+1)}{P(n)} = \frac{365-(n-1)}{365} \times \frac{n}{n-1}$$

$$\frac{P(n+1)}{P(n)} = \frac{(366-n)n}{365(n-1)}$$

Setting the ratio > 1:

$$(366-n)n > 365(n-1)$$

$$366n - n^2 > 365n - 365$$

$$n^2 - n - 365 < 0$$

Now, the zeroes of the quadratic $f(t) = t^2 - t - 365$ are $\frac{1 \pm \sqrt{1+4(365)}}{2} = \frac{1 \pm \sqrt{1461}}{2}$. Now, $\frac{1+\sqrt{1461}}{2} \approx \frac{1+38.22}{2} \approx 19.61$ and $\frac{1-\sqrt{1461}}{2} < 0$.

Hence, the ratio $\frac{P(n+1)}{P(n)}$ is greater than 1 for $n \leq 19$, meaning $P(20) > P(19)$. The ratio becomes less than 1 for $n \geq 20$, meaning $P(21) < P(20)$.

Correct option: **(B)**.

MCQ 2

Consider a linear regression model with n observations and p covariates. We compare two different regularization approaches with a tuning parameter $\lambda > 0$:

Model 1 (Bridge Regression): A regression fit with an L_q penalty where $1 < q < 2$. Let the fitted parameter vector be $\hat{\beta}_1$.

Model 2 (Elastic Net): An Elastic Net fit with a mixing parameter $\alpha = 2 - q$. The penalty is defined as $\lambda\,[\alpha\|\beta\|_1 + (1-\alpha)\|\beta\|_2^2]$.

Let the fitted parameter vector be $\hat{\beta}_2$.Let $L(\lambda) = \|\hat{\beta}_1\|_0 - \|\hat{\beta}_2\|_0$, where $\|\cdot\|_0$ denotes the L_0 norm (the number of non-zero components in the vector).Which of the following statements are correct? (More than 1 correct type)

(A) $L(\lambda)$ is a non-decreasing function of λ.

(B) $\|\hat{\beta}_1 - \hat{\beta}_2\|_2 \to 0$ as $\lambda \to \infty$.

(C) $\|\hat{\beta}_1 - \hat{\beta}_2\|_0 \to 0$ as $\lambda \to \infty$.

(D) The maximum possible value of $L(\lambda)$ is $p - 1$.

Solution. **Option (A): Correct.** Bridge has a L_q penalty which is differentiable so Bridge does not perform model selection, but elastic net does, so as lambda increases the L_0 norm (which is just the number of nonzero parameters), decreases for Elastic Net but remains same for Bridge. So $L(\lambda)$ is non-decreasing.

Option (B): Correct. As $\lambda \to \infty$, both the models converge towards an intercept only model.

Option (C) Incorrect. Bridge never zeroes out a coefficient, Elastic Net does. (Again L_0 norm is just the number of non-zero components of a vector).

Option (D) Incorrect. The maxima is p, same logic as option (C).

MCQ 3

3 points are randomly chosen on the circumference of a circle of radius r, what is the probability that the triangle formed by joining them is acute?

(A) $\frac{1}{4}$

(B) $\frac{3}{4}$

(C) $\frac{1}{2}$

(D) Will depend on the value of r

Solution. Let three points P_1, P_2, P_3 be chosen uniformly and independently on the circumference of a circle of length $L = 1$. Let the positions be $0, x, y$ where $x, y \in [0, 1]$.

Geometric Condition: A triangle is acute if and only if no arc between any two points exceeds a semicircle:

$$\text{Arc}_1 < \frac{1}{2}, \quad \text{Arc}_2 < \frac{1}{2}, \quad \text{Arc}_3 < \frac{1}{2}$$

Case Analysis: $(x < y)$ The lengths of the arcs are x, $y - x$, and $1 - y$. The conditions become:

$$x < \frac{1}{2}, \quad y - x < \frac{1}{2}, \quad 1 - y < \frac{1}{2}$$

This simplifies to the system:

$$0 < x < \frac{1}{2} < y < x + \frac{1}{2}$$

Integration and Symmetry: The area A of this region in the unit square (where $x < y$) is:

$$A = \int_0^{1/2} \int_{1/2}^{x+1/2} dy\, dx = \int_0^{1/2} x\, dx = \frac{1}{8}$$

By symmetry, the case $y < x$ also yields an area of $\frac{1}{8}$.

The total area for an acute triangle is $P = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$. Since the radius r scales all lengths equally, it cancels out.

Answer: (A) $\frac{1}{4}$

MCQ 4

You are playing a one-player game with two opaque boxes. At each turn, you can choose either Place or Take.

- Place puts 1 from a third party into one of the two boxes, chosen uniformly at random.
- Take empties one of the two boxes, chosen uniformly at random, and the money in that box becomes yours.

The game consists of 100 turns, and at each turn you must choose either Place or Take. Assuming optimal play, if E is the expected payoff of this game, then report $\lfloor E \rfloor$.

Note that you do not know how much money you have taken until the end of the game.

(A) 49

(B) 62

(C) 73

(D) 92

Solution. Let x be the expected value in a single box.

- **Place:** Adds 0.5 to x (since box chosen uniformly).
- **Take:** Yields payoff x, then updates expected value to $x/2$.

The optimal strategy is to build value with $(100 - k)$ "Place" moves, then harvest with k "Take" moves.

1. Accumulation Phase: After $100 - k$ turns of "Place", the expected value per box is:

$$x_{\text{start}} = (100 - k) \times 0.5 = \frac{100 - k}{2}$$

2. Harvesting Phase: The total payoff is the sum of a geometric series with ratio 1/2:

$$E(k) = x_{\text{start}} + \frac{x_{\text{start}}}{2} + \cdots + \frac{x_{\text{start}}}{2^{k-1}} = x_{\text{start}} \left(\frac{1 - (1/2)^k}{1/2} \right) = 2x_{\text{start}}(1 - 2^{-k})$$

Substituting x_{start}:

$$E(k) = (100 - k)(1 - 2^{-k})$$

3. Maximization: We maximize $f(k) = (100 - k)(1 - 2^{-k})$ for integer k:

- $k = 5: \quad 95(1 - 1/32) \approx 92.03$
- $k = 6: \quad 94(1 - 1/64) = 94(63/64) = 92.53125$
- $k = 7: \quad 93(1 - 1/128) \approx 92.27$

Max value is ≈ 92.53. The integer part is:

$$\lfloor E \rfloor = \mathbf{92}.$$

MCQ 5

Let $X_1, X_2, \ldots, X_n$ be i.i.d. observations from a $N(\mu, \sigma^2)$ distribution, where σ^2 is known. To test the hypothesis

$H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$

we use the test statistic:

$$T(X) = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

For a given $\alpha \in (0, 1)$ and a weight parameter $q \in (0, 1)$, the decision rule is to reject H_0 if:

$$T(X) > \Phi^{-1}(1 - \alpha q) \quad \text{or} \quad T(X) < \Phi^{-1}(\alpha(1 - q))$$

where $\Phi(\cdot)$ denotes the Cumulative Distribution Function (CDF) of the standard normal distribution. Let $f(q)$ be the p-value of the observed data calculated as a function of $q \in (0, 1)$.Which of the following statements are correct? (More than 1 correct type)

(A) $f(q)$ is a piecewise linear and non-differentiable function in $(0, 1)$.

(B) $f(q)$ is differentiable everywhere in the interval $(0, 1)$.

(C) $f(q)$ attains a unique maximum in the interval $(0, 1)$.

(D) If q is a random variable such that $q \sim U(0, 1)$, then:

$$E[f(q)] = -\Phi(T)\ln(\Phi(T)) - (1 - \Phi(T))\ln(1 - \Phi(T))$$

Solution. **Deriving $f(q)$:** For an observed statistic t, the p-value $f(q)$ is the smallest α that would lead to rejection. Solving the boundary conditions

$$t = \Phi^{-1}(1 - \alpha q) \quad \text{and} \quad t = \Phi^{-1}(\alpha(1 - q))$$

for α, we find

$$f(q) = \min\left(1, \ \frac{1 - \Phi(t)}{q}, \ \frac{\Phi(t)}{1 - q}\right).$$

Option (A) & (B): Incorrect. The function is composed of reciprocal terms $\big(1/q\big)$ and $\big(1/(1-q)\big)$, so it is *not linear*. It has a "kink" (a point where it is not differentiable) at the intersection of the two curves, specifically at

$$q = 1 - \Phi(t).$$

Option C: Correct. The function increases up to $q = 1 - \Phi(t)$, where it attains its maximum value (which is 1), and then decreases, making the maximum unique.

Option D: Correct. Integrating $f(q)$ over $q \in [0, 1]$ involves logarithmic integrals of the form

$$\int \frac{c}{x}\, dx.$$

Evaluating this integral leads exactly to the entropy-like expression

$$-P \ln P - (1 - P)\ln(1 - P), \quad \text{where } P = \Phi(T).$$

MCQ 6

Let $V = \mathbb{C}^d$ be a d-dimensional inner product space with the standard inner product. Two orthonormal bases $\mathcal{B} = \{e_1, \ldots, e_d\}$ and $\mathcal{C} = \{f_1, \ldots, f_d\}$ are said to be Mutually Unbiased Bases (MUB) if:

$$|\langle e_i, f_j \rangle|^2 = \frac{1}{d} \quad \text{for all } i, j.$$

Fix $i \in \{1, \ldots, d\}$ and write:

$$e_i = \sum_{j=1}^{d} a_j f_j, \quad \text{where } a_j = \langle e_i, f_j \rangle.$$

Assume that $\mathcal{B}$ and $\mathcal{C}$ are mutually unbiased. Which of the following statements is necessarily true?

(A) $\sum_{j=1}^{d} a_j = 0$

(B) $\sum_{j=1}^{d} |a_j|^4 = \frac{1}{d}$

(C) $\sum_{j=1}^{d} |a_j|^4 = \frac{1}{d^2}$

(D) $\sum_{j=1}^{d} a_j^2 = 1$

Solution. Since $\mathcal{B}$ and $\mathcal{C}$ are mutually unbiased,

$$|a_j|^2 = |\langle e_i, f_j \rangle|^2 = \frac{1}{d} \quad \text{for all } j.$$

Therefore,

$$\sum_{j=1}^{d} |a_j|^4 = d\left(\frac{1}{d}\right)^2 = \frac{1}{d}.$$

The other options depend on the phases of a_j and are not determined by the MUB condition.

Correct answer: (B)

MCQ 7

Let $V = \mathbb{C}^d$ be a d-dimensional inner product space with the standard inner product. Two orthonormal bases $\mathcal{B} = \{e_1, \dots, e_d\}$ and $\mathcal{C} = \{f_1, \dots, f_d\}$ are said to be Mutually Unbiased Bases (MUB) if:

$$|\langle e_i, f_j \rangle|^2 = \frac{1}{d} \quad \text{for all } i, j.$$

For each i, j, define the rank-one projection matrices:

$$P_i = e_i e_i^*, \quad Q_j = f_j f_j^*$$

where vectors are viewed as column vectors and $(\cdot)^*$ denotes the conjugate transpose. Which of the following conditions is equivalent to the statement that $\mathcal{B}$ and $\mathcal{C}$ are mutually unbiased?

(A) $\mathrm{Tr}(P_i Q_j) = 0$ for all $i \neq j$

(B) $\mathrm{Tr}(P_i Q_j) = \frac{1}{d}$ for all i, j

(C) $P_i Q_j = Q_j P_i$ for all i, j

(D) $\sum_{i=1}^{d} P_i Q_i = I$

Solution. We compute

$$\mathrm{Tr}(P_i Q_j) = \mathrm{Tr}(e_i e_i^* f_j f_j^*) = |\langle e_i, f_j \rangle|^2.$$

Hence the condition

$$\mathrm{Tr}(P_i Q_j) = \frac{1}{d} \quad \forall i, j$$

is equivalent to the definition of mutually unbiased bases.

Correct answer: (B)

MCQ 8

Let $V = \mathbb{C}^d$ be a d-dimensional inner product space with the standard inner product. Two orthonormal bases $\mathcal{B} = \{e_1, \ldots, e_d\}$ and $\mathcal{C} = \{f_1, \ldots, f_d\}$ are said to be Mutually Unbiased Bases (MUB) if:

$$|\langle e_i, f_j \rangle|^2 = \frac{1}{d} \quad \text{for all } i, j.$$

Let $\mathcal{B}$ and $\mathcal{C}$ be orthonormal bases of $\mathbb{C}^d$ and define:

$$M(\mathcal{B}, \mathcal{C}) := \max_{i,j} |\langle e_i, f_j \rangle|^2$$

Which of the following statements is correct?

(A) $M(\mathcal{B}, \mathcal{C}) \geq \frac{1}{d}$, with equality if and only if $\mathcal{B}$ and $\mathcal{C}$ are mutually unbiased.

(B) $M(\mathcal{B}, \mathcal{C}) \leq \frac{1}{d}$ for all orthonormal bases.

(C) $M(\mathcal{B}, \mathcal{C}) = 1$ if and only if the bases are mutually unbiased.

(D) $M(\mathcal{B}, \mathcal{C})$ is independent of d.

Solution. For fixed i, orthonormality gives

$$\sum_{j=1}^{d} |\langle e_i, f_j \rangle|^2 = 1.$$

Thus,

$$\max_j |\langle e_i, f_j \rangle|^2 \geq \frac{1}{d}.$$

Taking the maximum over all i, j yields

$$M(\mathcal{B}, \mathcal{C}) \geq \frac{1}{d}.$$

Equality holds if and only if all overlaps equal $1/d$, i.e., $\mathcal{B}$ and $\mathcal{C}$ are mutually unbiased.

Correct answer: (A)

MCQ 9

There are 2026 distinct balls kept in a single line. In a given move 2 balls are randomly chosen and swapped. What is the probability that after 2026 moves, the balls will be one swap away from their original orientation?

(A) $\frac{1}{2}$

(B) $\frac{571}{2026}$

(C) $\frac{1}{2026}$

(D) 0

Solution. Initially, the balls are in their original order, which corresponds to the identity permutation. Each move consists of swapping two balls, i.e., performing a *transposition*.

A transposition is an *odd permutation*. Hence, every swap changes the parity of the permutation.

After 2026 swaps, since 2026 is even, the resulting permutation must be an *even permutation*.

Now, being *one swap away from the original orientation* means that the current arrangement can be transformed back to the identity permutation using exactly one transposition. This implies that the current permutation itself is a transposition, which is an *odd permutation*.

Thus, we arrive at a contradiction:

- After 2026 swaps, the permutation must be even.
- Being one swap away from the identity requires the permutation to be odd.

Since a permutation cannot be both even and odd, such a configuration is impossible.

Therefore, the required probability is 0.

MCQ 10

A proprietary trading desk operates a binary daily trading strategy on a stock index. Each trading day results in one of two possible outcomes: a profitable day, denoted by W, or a losing day, denoted by L.

Over a historical backtest covering 100 trading days, the strategy records a total of 60 profitable days and 40 losing days. It is assumed that the sequence of wins and losses is generated in a random order, and that there is no serial dependence in the outcomes, in particular, the strategy exhibits neither momentum nor mean-reversion effects across consecutive days.

A run is defined as a maximal sequence of identical outcomes occurring on consecutive trading days. For example, a sequence of three profitable days in a row (WWW) constitutes a single run, while an alternating sequence such as WLW consists of three runs.

Under these assumptions, what is the expected number of runs in the 100-day trading record?

(A) 51

(B) 49

(C) 50

(D) 77

Solution. Let the total number of trading days be $n = 100$, with $n_W = 60$ profitable days (W) and $n_L = 40$ losing days (L).

The outcomes are assumed to be randomly ordered with no serial dependence.

A *run* is defined as a maximal sequence of identical outcomes occurring consecutively.

To calculate the expected number of runs, define indicator random variables

$$I_i = \begin{cases} 1, & \text{if day } i \text{ is different from day } i-1, \\ 0, & \text{otherwise,} \end{cases} \qquad i = 2, \ldots, n.$$

Then, the total number of runs R is

$$R = 1 + \sum_{i=1}^{n} I_i.$$

It's easy to check that $\mathbb{E}[I_i] = \dfrac{2\, n_W\, n_L}{n(n-1)}$.

For a sequence consisting of two symbols arranged uniformly at random, the expected number of runs is given by

$$\mathbb{E}[\text{Number of runs}] = 1 + \frac{2n_W n_L}{n}.$$

Substituting the given values,

$$\mathbb{E}[\text{Number of runs}] = 1 + \frac{2 \times 60 \times 40}{100} = 1 + \frac{4800}{100} = 49.$$

Therefore, the expected number of runs is $\boxed{49}$. **Correct option: (B)**

MCQ 11

In Principal Component Analysis (PCA), we seek a direction vector $u \in \mathbb{R}^p$ that maximizes the variance of the projected data. If Σ is the sample covariance matrix, this is formulated as:

$$\text{Maximize } u^T \Sigma u \quad \text{subject to } u^T u = 1$$

Using the method of Lagrange Multipliers, let $\mathcal{L}(u, \lambda) = u^T \Sigma u - \lambda(u^T u - 1)$. Which of the following statements are correct?

(A) The stationary points of $\mathcal{L}$ satisfy the equation $\Sigma u = \lambda u$.

(B) The maximum variance is equal to the largest eigenvalue of Σ.

(C) If Σ is a rank-deficient matrix, there is no solution to this optimization problem.

(D) The Lagrange multiplier λ represents the variance of the data projected onto the direction u

Solution. We wish to maximize $u^T \Sigma u$ subject to the constraint $u^T u = 1$. The Lagrangian is

$$\mathcal{L}(u, \lambda) = u^T \Sigma u - \lambda(u^T u - 1).$$

Taking gradient with respect to u and setting it to zero,

$$\nabla_u \mathcal{L} = 2\Sigma u - 2\lambda u = 0 \quad \Rightarrow \quad \Sigma u = \lambda u.$$

Thus, stationary points are eigenvectors of Σ, proving (A).

At a stationary point,

$$u^T \Sigma u = u^T (\lambda u) = \lambda(u^T u) = \lambda,$$

so the variance equals the corresponding eigenvalue. Hence, the maximum variance is the largest eigenvalue of Σ, proving (B).

If Σ is rank-deficient, it still has eigenvectors and eigenvalues (possibly zero), and the optimization over the unit sphere still attains a maximum. Therefore, (C) is false.

Since $\lambda = u^T \Sigma u$, the Lagrange multiplier represents the variance of the data projected onto u, proving (D).

Correct options: (A), (B), (D)

MCQ 12

In training a neural network or fitting a generalized linear model, we often minimize a loss function $L(\theta)$ where $\theta \in \mathbb{R}^p$. Let $H(\theta) = \nabla^2 L(\theta)$ be the Hessian matrix of the loss function at a critical point θ^* (where $\nabla L(\theta^*) = 0$).Suppose $L(\theta)$ is twice continuously differentiable. Which of the following statements are correct?

(A) If $H(\theta^*)$ has at least one positive and one negative eigenvalue, θ^* is a saddle point.

(B) If $H(\theta^*)$ is positive definite, the Newton-Raphson update step $\Delta\theta = -H(\theta^*)^{-1}\nabla L(\theta)$ points in a descent direction.

(C) For a ridge regression objective $L(\beta) = \|y - X\beta\|_2^2 + \lambda\|\beta\|_2^2$, the Hessian $H(\beta)$ is independent of β.

(D) If $L(\theta)$ is a strictly convex function, then $H(\theta)$ must be singular for all θ.

Solution. Let θ^* be a critical point of $L(\theta)$, i.e., $\nabla L(\theta^*) = 0$, and let $H(\theta^*) = \nabla^2 L(\theta^*)$ denote the Hessian matrix.

(A) If $H(\theta^*)$ has at least one positive and one negative eigenvalue, then the second-order Taylor expansion shows that $L(\theta)$ increases in some directions and decreases in others around θ^*. Hence, θ^* is a saddle point. **(Correct)**

(B) The Newton–Raphson update is

$$\Delta\theta = -H(\theta^*)^{-1}\nabla L(\theta).$$

If $H(\theta^*)$ is positive definite, then $H(\theta^*)^{-1}$ is also positive definite, and

$$\nabla L(\theta)^T \Delta\theta = -\nabla L(\theta)^T H(\theta^*)^{-1}\nabla L(\theta) < 0$$

for any nonzero gradient. Therefore, the update direction is a descent direction. **(Correct)**

(C) For ridge regression,

$$L(\beta) = \|y - X\beta\|_2^2 + \lambda\|\beta\|_2^2.$$

The Hessian is

$$H(\beta) = 2X^T X + 2\lambda I,$$

which is independent of β. **(Correct)**

(D) If $L(\theta)$ is strictly convex, then its Hessian is positive definite (or positive semidefinite everywhere with strict convexity), and hence nonsingular. Therefore, the statement is false. **(Incorrect)**

Correct options: $(A), (B), (C)$

MCQ 13

What is the least number of colours required to color the integers $1, 2, ..., 2^n$ such that in any set of consecutive integers, there is a colour that occurs exactly once?

That is, the colouring has to satisfy the condition that for any integers i, j such that $1 \leq i \leq j \leq 2^n$, there is a colour that is given to exactly one integer in the set $i, i+1, ..., j-1, j$.

(A) $n+1$

(B) n

(C) $n-1$

(D) $\frac{n}{2}$

Solution. We wish to color the integers $1, 2, \ldots, 2^n$ so that in every consecutive set

$$\{i, i+1, \ldots, j\}, \quad 1 \leq i \leq j \leq 2^n,$$

there exists a color that occurs exactly once.

Construction (Upper Bound):

For each integer k, write

$$k = 2^t m,$$

where m is odd, and define the color of k to be t, the largest power of 2 dividing k. This assigns colors $0, 1, 2, \ldots, n$, so $n+1$ colors are used.

Consider any interval $[i, j]$. Among the integers in this interval, choose the one divisible by the largest power of 2. This integer is unique in the interval, since two distinct integers in a consecutive interval cannot both be divisible by the same highest power of 2. Hence, the corresponding color appears exactly once in the interval.

Thus, a coloring using $n+1$ colors satisfying the required condition exists.

Lower Bound:

Suppose fewer than $n+1$ colors are used. By the pigeonhole principle applied to intervals of length 2^k, for some interval all colors must appear either zero times or at least twice, contradicting the given condition.

Therefore, at least $n+1$ colors are necessary.

Conclusion:

The least number of colors required is

$$\boxed{n+1}.$$

Correct option: (A)

MCQ 14

Let X and Y be two independent standard normal random variables, $X, Y \sim N(0,1)$. We apply the following transformation to represent them in polar coordinates (R, Θ):

$X = R\cos\Theta \; Y = R\sin\Theta$

where $R \in [0, \infty)$ and $\Theta \in [0, 2\pi)$. Let J be the Jacobian matrix of the transformation from (R, Θ) to (X, Y). Which of the following is true regarding the absolute value of the determinant of the Jacobian, $|\det(J)|$, and the resulting joint density?

(A) $|\det(J)| = R^2$

(B) $|\det(J)| = R$

(C) The joint density of (R, Θ) is independent of Θ, implying Θ is uniformly distributed on $[0, 2\pi)$.

(D) The transformation is non-invertible at the origin, so the Jacobian is undefined for all $R > 0$.

Solution. The transformation from polar to Cartesian coordinates is given by

$$X = R\cos\Theta, \qquad Y = R\sin\Theta,$$

with $R \geq 0$ and $\Theta \in [0, 2\pi)$.

Jacobian Determinant:

The Jacobian matrix of the transformation is

$$J = \begin{pmatrix} \frac{\partial X}{\partial R} & \frac{\partial X}{\partial \Theta} \\ \frac{\partial Y}{\partial R} & \frac{\partial Y}{\partial \Theta} \end{pmatrix} = \begin{pmatrix} \cos\Theta & -R\sin\Theta \\ \sin\Theta & R\cos\Theta \end{pmatrix}.$$

The determinant is

$$\det(J) = R(\cos^2\Theta + \sin^2\Theta) = R.$$

Hence,

$$|\det(J)| = R,$$

so statement (B) is correct, while (A) is false.

Joint Density of (R, Θ):

The joint density of (X, Y) is

$$f_{X,Y}(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}.$$

Using $x^2 + y^2 = R^2$ and the Jacobian,

$$f_{R,\Theta}(r, \theta) = f_{X,Y}(r\cos\theta, r\sin\theta)\,|\det(J)| = \frac{1}{2\pi} e^{-r^2/2}\, r, \quad r \geq 0,\ \theta \in [0, 2\pi).$$

This density does not depend on θ, implying that Θ is uniformly distributed on $[0, 2\pi)$ and independent of R. Thus, statement (C) is correct.

The transformation is non-invertible only at $R = 0$, but the Jacobian is well-defined for all $R > 0$, so statement (D) is false.

Correct options: (B) and (C)

MCQ 15

The sequence given by $x_0 = a$, $x_1 = b$, and

$$x_{n+1} = \frac{1}{2}\left(x_{n-1} + \frac{1}{x_n}\right)$$

is periodic.

The value of ab is

(A) -1

(B) $\frac{1}{2}$

(C) Cannot be determined using the given information.

(D) None of these.

Solution. Let the given recurrence relation be:

$$x_{n+1} = \frac{1}{2}\left(x_{n-1} + \frac{1}{x_n}\right) \tag{1}$$

Multiplying both sides by x_n, we get:

$$x_n x_{n+1} = \frac{1}{2} x_n x_{n-1} + \frac{1}{2}$$

Let $u_n = x_n x_{n+1}$. The equation becomes a linear recurrence relation for u_n:

$$u_n = \frac{1}{2} u_{n-1} + \frac{1}{2}$$

Subtracting 1 from both sides:

$$u_n - 1 = \frac{1}{2}(u_{n-1} - 1)$$

This implies that $u_n - 1$ follows a geometric progression with ratio $\frac{1}{2}$. The general solution is:

$$u_n - 1 = \frac{1}{2^n}(u_0 - 1) \implies u_n = 1 + \frac{u_0 - 1}{2^n}$$

For the sequence x_n to be periodic, the product of consecutive terms u_n must also be periodic. However, the term $\frac{u_0-1}{2^n}$ strictly decays to 0 as $n \to \infty$ unless the coefficient is zero. Thus, for u_n to be periodic (specifically, constant), we must have:

$$u_0 - 1 = 0 \implies u_0 = 1$$

Since $u_0 = x_0 x_1 = ab$, we have:

$$ab = 1$$

Checking the options: (A) -1, (B) $\frac{1}{2}$. Since 1 is not listed, the correct choice is "None of these".

Answer: (D) None of these

MCQ 16

Consider the following game played on an infinite grid with square cells (like an infinite chessboard in which all squares are white). First, some n cells that are chosen arbitrarily are coloured black. The game now begins. In each round, some cells coloured black change to white, some cells coloured white change to black and other cells remain the same according to the following rule: the colour of a cell is toggled if and only if its current colour is different from the colour of the cell to its right and the colour of the cell above it. After how many rounds will every cell necessarily become white?

(A) $n+1$

(B) n

(C) $n-1$

(D) $\frac{n}{2}$

Solution. Let $c(x, y, t) \in \{0, 1\}$ denote the color of the cell at (x, y) at time t (1 = Black, 0 = White). Let R be the neighbor to the right $(x+1, y)$ and A be the neighbor above $(x, y+1)$.

The given update rule is: a cell toggles if $c \neq R$ and $c \neq A$.

- If $c = 1$: It becomes 0 if $R = 0$ and $A = 0$. Else it stays 1.
- If $c = 0$: It becomes 1 if $R = 1$ and $A = 1$. Else it stays 0.

This implies a black cell survives only if it is "supported" by a black cell to its right or above.

Worst-Case Scenario Analysis

Consider a configuration of n black cells arranged in a horizontal row: $(1, 0), (2, 0), \ldots, (n, 0)$.

1. The rightmost cell $(n, 0)$ has $R = 0$ and $A = 0$. It turns white at $t = 1$.
2. The cell $(n-1, 0)$ has $R = 1$ initially, so it survives the first update. However, at $t = 1$, its right neighbor becomes 0. Consequently, $(n-1, 0)$ turns white at $t = 2$.
3. By induction, the black strip erodes one cell at a time from the right.

Thus, it takes exactly n rounds for the last cell at $(1, 0)$ to turn white. No other configuration of size n can delay the "erosion" longer than this linear dependency.

Answer: The cells will necessarily become white after $\boxed{n}$ rounds.

Correct option: (B)

MCQ 17

Suppose we are given a training dataset $\{(x_i, y_i)\}_{i=1}^n$ where $x_i \in \mathbb{R}^p$ and labels $y_i \in \{+1, -1\}$. We fit two binary classification models by minimizing the following objective functions:

Model 1 (Logistic Regression):

$$\min_\beta \sum_{i=1}^n \ln(1 + \exp(-y_i x_i^T \beta)) + \frac{\lambda}{2}\|\beta\|_2^2$$

Model 2 (Soft-Margin SVM):

$$\min_\beta \sum_{i=1}^n \max(0, 1 - y_i x_i^T \beta) + \frac{\lambda}{2}\|\beta\|_2^2$$

We initially fit both models on the n points. We then add 1,000 new observations to the training set. It is observed that for every new point (x_{new}, y_{new}), the condition $y_{new} x_{new}^T \hat{\beta} > 1$ holds for both the current optimal Logistic Regression estimate and the current SVM estimate. Which of the following statements are correct?

(A) For a small λ, adding these 1,000 points changes the optimal β for the SVM model, but not for the Logistic Regression model.

(B) For a small λ, adding these 1,000 points changes the optimal β for the Logistic Regression model, but not for the SVM model.

(C) The magnitude of the shift in the fitted parameter vector β (for either model) decreases as the regularization parameter λ is increased.

(D) The magnitude of the shift in the fitted parameter vector β (for either model) increases as the regularization parameter λ is increased.

Solution. Option **B** is correct, option **A** is not. Observe the given condition implies that the new observations don't even alter the error of the SVM model but they do affect the error of the Logistic Model. (This is a standard property of SVM, the decision boundary only depends on the Support Vectors)

Option **C** is correct, option **D** is not. As λ increases, the penalty term dominates the objective error term so the coefficients are shrunk towards 0. In this highly regularized state, the influence of individual data points (the "data term") on the final parameters becomes negligible.

Hunter Spider

Arachnia, a hungry spider is looking for a prey in a narrow stick, while Fling, a fly nearby has seen the predator approaching. Arachnia has still not seen any fly, so it is jumping randomly. The stick is narrow enough that the spider can only move back and forth along one line only, assume it to be the X-axis, with the current position of the spider as origin.

Also assume that the spider jumps along positive X direction with probability p and negative X direction with probability $1 - p$, and the magnitude of the jump is distributed as Uniform(0,l), where $0 < p < 1$, $l > 0$ are unknown parameters, all the jumps are independent and direction and magnitude of each jump are independent. The fly looks at the position of the spider for 10 consecutive jumps, and decides to go as far as possible from the position of the spider after the next jump. The fly tries to get a reasonable estimate as fast as possible - Calculate the MLEs of p, l assume them to be the true values of the parameters and calculate the expectation of the position after the next jump as an estimate.

Your answer will be considered correct if the absolute or relative error does not exceed 10^{-6}.

Input Format: The input consists of 10 integers, each given on a separate line, representing the positions of the spider after 10 consecutive jumps.

Output Format: The predicted position after next jump.

Sample 1:

Input	Output
3	7.0
4	
1	
-2	
8	
4	
6	
5	
4	
7	

Sample 2:

Input	Output
1.2	9.667000000000002
2.1	
3.8	
2.6	
4.89	
5.62	
6.99	
7.81	
9.21	
8.98	

```python
def indicator_positive(x):
    return 1 if x > 0 else 0
def spider(ls):
    b=[abs(ls[0]),0,0,0,0,0,0,0,0,0]
    c=indicator_positive(ls[0])
    for i in range(1,10):
        c=c+indicator_positive(ls[i]-ls[i-1])
        b.append(abs(ls[i]-ls[i-1]))
    L=max(b)
    p=c/10
    print(b)
    print(p)
    return (ls[9]+p*L/2-(1-p)*L/2)
ls=[]
for i in range(10):
    st='enter the position at '+ str(i+1) + 'th time '
    a=float(input(st))
    ls.append(a)
print('the expected next position is ', spider(ls))
```

Red Circle, Blue Circle

There are $3N$ positive integers (not necessarily distinct) written on a board. Alice and Bob play a game with N rounds. In each round, Alice circles a number with red colour while Bob circles a number with blue colour and erases some number that has not been coloured earlier. After N rounds, there are N numbers in red circles, and N numbers in blue circles. Alice wins if the sum of the numbers in red circles is not equal to the sum of the numbers in blue circles, while Bob wins if the two sums are equal.

Input Format: The first line contains an integer N.

The second line contains $3N$ integers (separated by whitespace character)

Output Format: Assuming both Alice and Bob play optimally, output whether Alice has a winning strategy or not as Y/N .

Constraints: $0 \leq N \leq 10^6$

Sample 1:

Input	Output
4 1 1 1 1 1 1 1 1 1 1 1 1	N

Sample 2:

Input	Output
2 1 2 4 8 16 32	Y

```python
from collections import Counter
def solve(n, nums):
    freq = Counter(nums)
    for cnt in freq.values():
        if cnt % 3 != 0:
            print('Y')
            return
    print('N')
if __name__ == "__main__":
    n = int(input().strip())
    nums = list(map(int, input().split()))
    solve(n, nums)
```

Scam Alert

The Scam Intelligence Agency (SIA) of a city has information about an ongoing scam in the neighboring cities. A foreigner comes to the city and selects a person and asks him for 100 Rs and tells that person to ask 100 Rs from two of his friends and to tell them to ask two of their friends and so on. If every body manages to get the money from two friends then everybody is at profit. SIA immediately identifies this as a scam.

Given any person from the population, SIA knows which two people they will ask for money. SIA also knows that the people in the city are susceptible to the scam only once. Once they give money, they are no longer going to give or take money anymore. For example, in the picture below we depict a town with five people.

An arrow from A to B indicates that A would ask B for the money. In this example, B can lose money. We can check that with the following scenario.

1. Someone from outside the town asks A for money.
2. A asks B for money.
3. A asks C for money.
4. C asks D for money.
5. B asks C for money.
6. B asks D for money.

Observe that when B asks C and D for money, they will not give it to B since they have already given money to someone else.

Your answer will be considered correct if the absolute or relative error does not exceed 10^{-6}.

Input Format: The first line contains an integer N ($3 \leq N \leq 1000$) indicating the number of people in the city. Each person is identified by a distinct integer from 1 to N. For $i = 1, 2, \ldots, N$, the i-th of the next N lines contains two integers X_i and Y_i ($1 \leq X_i, Y_i \leq N, X_i, Y_i 6 = i$ and $X_i 6 = Y_i$), representing that person i would ask for money to person X_i and person Y_i

Output Format: Output a single line with a string of length N such that its i-th character is the uppercase letter "Y" if person i can lose money, and the uppercase letter "N" otherwise.

Constraints: $3 \leq n$

Sample 1:

Input	Output
5	YYYYY
2 3	
3 4	
4 5	
5 1	
1 2	

Sample 2:

Input	Output
4	NYYY
2 3	
3 4	
2 4	
2 3	

Sample 3:

Input	Output
4	YYYN
2 3	
3 1	
1 2	
2 3	

```python
import numpy as np
n=int(input())
A = [[0]*n for _ in range(n)]
i=0
while i<n:
    l, r = map(int, input().split())
    l-=1
    r-=1
    if(l!=i and r!=i and l!=r):
        A[i][l]=1
        A[i][r]=1
        i+=1
    else:
        print("Invalid input, Re-enter")
col_counts = [sum(col) for col in zip(*A)]
result=[]
for i in range(n):
    if (col_counts[i] == 0 or any(col_counts[j] == 1 for j in range(n) if A[i][j] == 1)):
        result.append('N')
    else:
        result.append('Y')
output_string = ''.join(result)
print(output_string)
```